§17.5 Divergence and Curl

Def. ∇ is called the <u>del operator</u>.

In $\mathbb{R}^2$, $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$. In $\mathbb{R}^3$, $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. And so on.

We already know the gradient of a scalar-valued function is grad $f = \nabla f$. $\mathbb{R}^2$: $\nabla f = \langle f_x, f_y \rangle$, $\mathbb{R}^3$: $\nabla f = \langle f_x, f_y, f_z \rangle$.

<u>Divergence</u> measures the expansion or contraction of a vector field and is defined $\boxed{\text{div } \vec{F} = \nabla \bullet \vec{F}}$

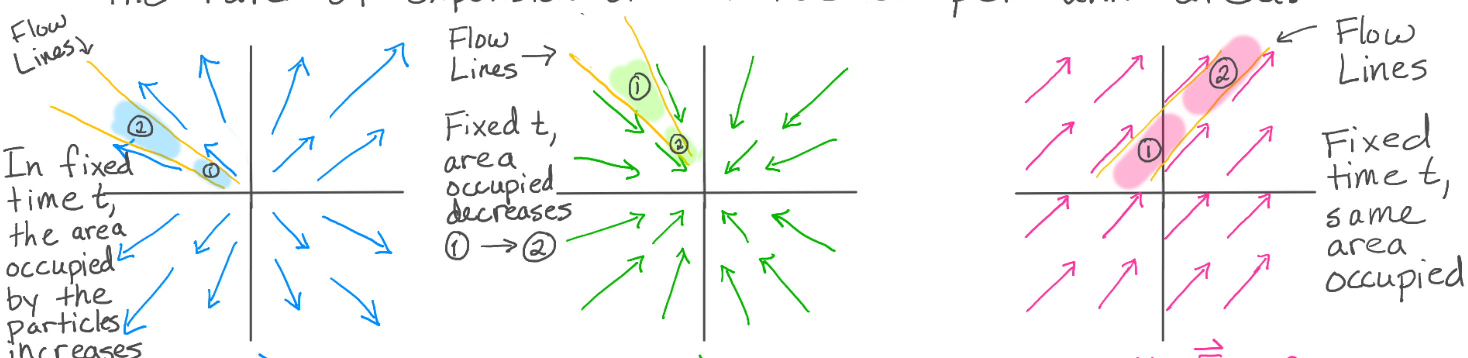
If $\vec{F} = \langle f, g \rangle$, div $\vec{F} = \nabla \bullet \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \bullet \langle f, g \rangle = f_x + g_y$.

If $\vec{F} = \langle f, g, h \rangle$, div $\vec{F} = \nabla \bullet \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \bullet \langle f, g, h \rangle = f_x + g_y + h_z$.

What does this mean?

If 3D, if $\vec{F}$ is the velocity field of a gas or liquid (meaning for a particle at P, $\vec{F}(P)$ is the velocity of the gas or liquid particle. The divergence of $\vec{F}$ measures the rate of expansion or contraction per unit volume under the flow of the gas or liquid. If div $\vec{F} > 0$, the gas or liquid is expanding, and if div $\vec{F} < 0$, the gas or liquid is contracting.

In 2D, if $\vec{F}$ is the velocity field, then divergence measures the rate of expansion or contraction per unit area.



div $\vec{F} > 0$
The lengths of the vectors are increasing away from the origin.

div $\vec{F} < 0$
The lengths of the vectors are decreasing towards the origin

div $\vec{F} = 0$
The lengths of the vectors stay the same away and towards the origin.

Def. If div $\vec{F} = 0$, the vector field $\vec{F}$ is <u>source free</u> (or divergence free). This means there is no expansion away from a source and no contraction towards a "source".

Thm Divergence of Radial Vector Fields

If $\vec{F}(x,y,z)$ can be written as $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$,

where $\vec{r} = \langle x, y, z\rangle$, then $\boxed{\operatorname{div}\vec{F} = \dfrac{3-p}{|\vec{r}|^p}}$.

"Radial" because $|\vec{r}| = \sqrt{x^2+y^2+z^2}$ is the radius of a sphere.

Ex. 1 Find the divergence of $\vec{F} = \langle xz, -y^2, xyz\rangle$.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[xz] + \frac{\partial}{\partial y}[-y^2] + \frac{\partial}{\partial z}[xyz]$$
$$= \boxed{z - 2y + xy} \quad \blacksquare$$

Def. The curl of a vector field measures the rotation of a vector field and is defined $\boxed{\operatorname{curl}\vec{F} = \nabla\times\vec{F}}$.

If $\vec{F} = \langle f, g\rangle$, we use the definition from §17.4 and have $\boxed{\operatorname{curl}\vec{F} = g_x - f_y}$.

If $\vec{F} = \langle f, g, h\rangle$, we use the formula $\nabla\times\vec{F}$ as follows

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \langle f, g, h\rangle$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, -\left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right), \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right\rangle$$
$$= \boxed{\langle h_y - g_z, f_z - h_x, g_x - f_y\rangle}$$

The physical interpretation of the curl is described in this figure from the book.

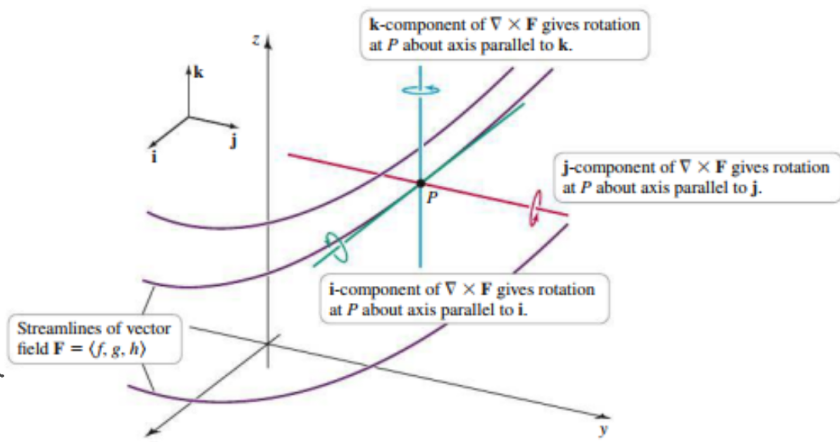


Figure 17.40

Def If $\operatorname{curl}\vec{F} = \vec{0}$, the vector field $\vec{F}$ is irrotational.

Ex.2 Find the curl of $\vec{F} = \langle xz, -y^2, xyz \rangle$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \times \langle xz, -y^2, xyz \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y^2 & xyz \end{vmatrix}$$

$$= \langle xz - 0, -(yz - x), 0 - 0 \rangle$$

$$= \boxed{\langle xz, x - yz, 0 \rangle} \quad \blacksquare$$

Ex.3 Show $\text{div}(f\vec{F}) = f \, \text{div} \vec{F} + \vec{F} \bullet \nabla f$.

Write $\vec{F} = \langle P, Q, R \rangle$ for the component functions.

$$\begin{aligned}
\text{div}(f\vec{F}) &= \text{div}(f \langle P, Q, R \rangle) \\
&= \text{div}(\langle fP, fQ, fR \rangle) \\
&= \tfrac{\partial}{\partial x}[fP] + \tfrac{\partial}{\partial y}[fQ] + \tfrac{\partial}{\partial z}[fR] \quad \rightarrow \text{all product rules} \\
&= (f_x P + \underline{f} P_x) + (f_y Q + \underline{f} Q_y) + (f_z R + \underline{f} R_z) \\
&= f(P_x + Q_y + R_z) + (f_x P + f_y Q + f_z R) \\
&= f\left(\underline{\tfrac{\partial}{\partial x}} \underline{P} + \underline{\tfrac{\partial}{\partial y}} \underline{Q} + \underline{\tfrac{\partial}{\partial z}} \underline{R}\right) + \left(\underline{\tfrac{\partial f}{\partial x}} \underline{P} + \underline{\tfrac{\partial f}{\partial y}} \underline{Q} + \underline{\tfrac{\partial f}{\partial z}} \underline{R}\right) \\
&= f(\nabla \bullet \vec{F}) + (\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z} \rangle \bullet \vec{F}) \\
&= f(\text{div} \vec{F}) + \nabla f \bullet \vec{F}
\end{aligned}$$

$\blacksquare$

Note:	Gradient $\text{grad } f = \nabla f$	Divergence $\text{div } \vec{F} = \nabla \bullet \vec{F}$	Curl $\text{curl } \vec{F} = \nabla \times \vec{F}$
input	function of several variables	vector field	vector field
output	vector field	function of several variables	vector field