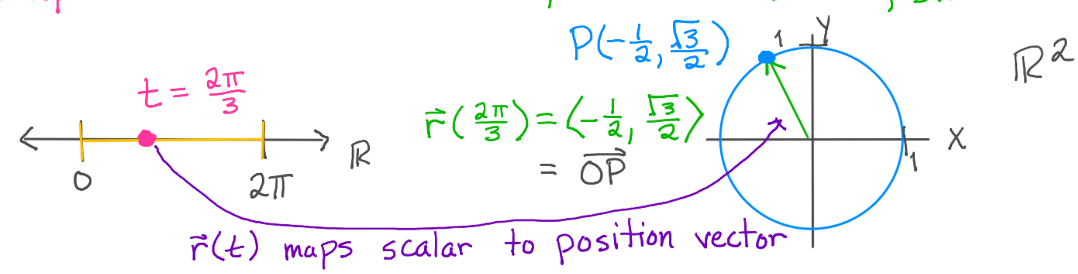
§17.6 Surface Integrals (Part 1)

Recall: The first method we learned to evaluate line integrals along a curve C was to parameterize the curve. In math, a curve is an object that has length but no width (i.e. we cannot find the area <u>of</u> a curve, but in 2D, we find the area below or between curves). Even though we can draw curves in $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$, all curves have the one dimension length and no width or volume. Because of this, we can take a curve in $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$ and describe it using a vector-valued function $\vec{r}(t)$ with $a \le t \le b$. $\vec{r}(t)$ has <u>one</u> scalar input and has the position vector for a point on the curve as output. <u>Parameterization</u> (or parametrization) is the process of describing a curve $f(x,y)=0$ or $f(x,y,z)=0$ in terms of a single <u>parameter</u> t (or $s, \theta, u, v, \dots$).

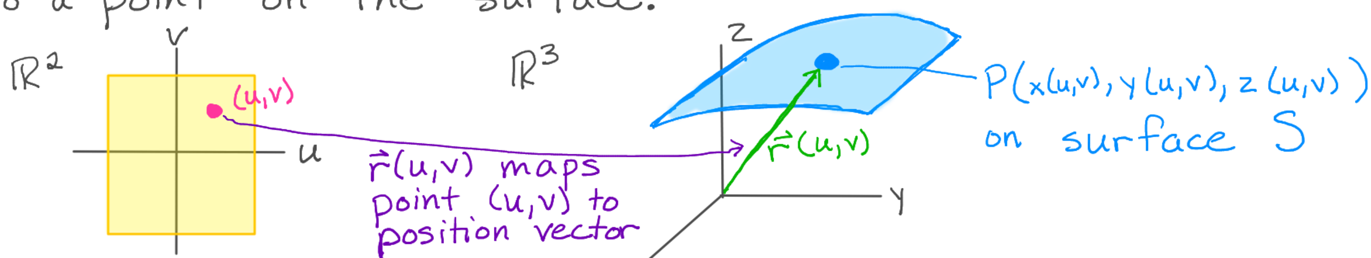
<u>Ex.</u> $x^2+y^2=1$ can be parameterized by $\vec{r}(t)=\langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$

Input: t Output: $\vec{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle$



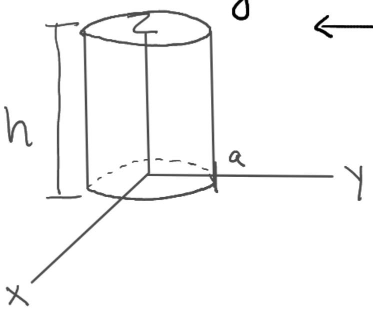
As t increases 0 to 2π, $\vec{r}(t)=\langle \cos t, \sin t\rangle$ outputs vectors that trace the curve $x^2+y^2=1$.

Now, we want to parameterize a surface. Because a surface has area, it should have some notion of "length" and "width." Meaning that to parameterize the surface, we will use <u>two</u> parameters ((u,v), (s,t), (r,θ) ...). The parameterization $\vec{r}(u,v)$ will map a point in $\mathbb{R}^2$ to a position vector (in $\mathbb{R}^2$ or $\mathbb{R}^3$) to a point on the surface.



Input: (u,v) Output: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Ex. 1 Give a parametric description of a cylinder of radius a and height h. Include bounds for the parameters.



$\leftarrow$ Use this figure

In rectangular coordinates, the cylinder is $\{(x,y,z) : x = a\cos\theta,\ y = a\sin\theta,\ z = z$ with $0 \le \theta \le 2\pi$ and $0 \le z \le h\}$.

Could parameterize with

$$\vec{r}(\theta, z) = \langle a\cos\theta,\ a\sin\theta,\ z\rangle,\quad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le z \le h \end{matrix}$$

OR

Basically cylindrical coordinates with $r = a$ fixed. $\leftarrow$

$$\vec{r}(u,v) = \langle a\cos u,\ a\sin u,\ v\rangle,\quad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le h \end{matrix}$$

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Ex. 2 Give a parametric description $\vec{r}(u,v)$ with $(u,v) \in R$ of the plane $2x - 4y + 3z = 16$.

We can solve the equation of the plane for any variable:

$x = \frac{1}{2}(16 + 4y - 3z)$ $\qquad$ $y = -\frac{1}{4}(16 - 2x - 3z)$ $\qquad$ $z = \frac{1}{3}(16 - 2x + 4y)$

$\downarrow$ $\qquad$ $\downarrow$ $\qquad$ $\downarrow$

$\vec{r}(u,v) = \langle \frac{1}{2}(16 + 4u - 3v),\ u,\ v\rangle$ $\qquad$ $\vec{r}(u,v) = \langle u,\ -\frac{1}{4}(16 - 2u - 3v),\ v\rangle$ $\qquad$ $\vec{r}(u,v) = \langle u,\ v,\ \frac{1}{3}(16 - 2u + 4v)\rangle$

(We usually go alphabetically, but $\vec{r}(u,v) = \langle \frac{1}{2}(16 + 4v - 3u),\ v,\ u\rangle$ is valid as well.)

Usually, we solve for z or whichever variable results in the most simplified coefficients.

Because a plane is unbounded, $R = \mathbb{R}^2$

$$= \{(u,v) : -\infty < u < \infty,\ -\infty < v < \infty\}.$$

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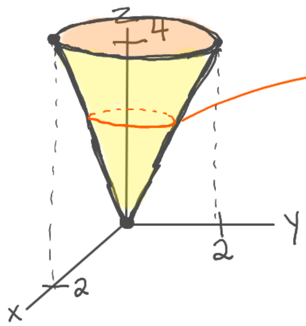
Ex.3 Give a parametric description $\vec{r}(u,v)$ with $(u,v) \in R$ for the surface of the cone $z^2 = 4(x^2+y^2)$, $0 \le z \le 4$.



On this circle: $z^2 = 4(x^2+y^2)$

$$\frac{z^2}{4} = x^2 + y^2$$

which means $x = \frac{z}{2}\cos\theta$ → use $\frac{z}{2}$ not $-\frac{z}{2}$ because $0 \le z \le 4$

$y = \frac{z}{2}\sin\theta$

$z = z$

and we can use $u = z$ and $v = \theta$ for the parameters

$$\boxed{\vec{r}(u,v) = \left\langle \frac{u}{2}\cos v,\ \frac{u}{2}\sin v,\ u \right\rangle,\ R = \{(u,v) : 0 \le u \le 4,\ 0 \le v \le 2\pi\}}$$

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Ex.4 Describe the surface with parametric representation $\vec{r}(u,v) = \langle u, u+v, 2-u-v \rangle$ for $0 \le u \le 2$, $0 \le v \le 2$.

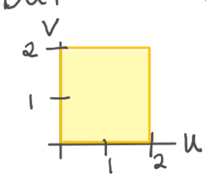
We have $x = u$, $y = u+v$, $z = 2-u-v$ so we can try to write an equation with $x, y,$ and z and no u, v.

$x = u$ and $y = u+v \Rightarrow y = x+v \Rightarrow v = y-x$

Then $z = 2-u-v \Rightarrow z = 2-x-(y-x) = 2-y$,

and the surface is $z = 2-y$ which is a plane,

but $0 \le u \le 2$ and $0 \le v \le 2$.



Since $x = u$, $0 \le x \le 2$.

Since $y = u+v$, the minimum value of y is 0 and the maximum value of y is 4 $(u=v=2)$.

$$\boxed{\text{The surface is the part of the plane } z = 2-y \text{ above } [0,2] \times [0,4] \text{ (in the xy-plane).}}$$

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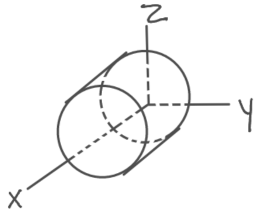
Ex.5 Describe the surface with parameterization
$\vec{r}(u,v) = \langle v, 6\cos u, 6\sin u\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 2$.

$x = v$
$y = 6\cos u$
$z = 6\sin u$

$> \; y^2 + z^2 = 36(\cos^2 u + \sin^2 u) = 36$



A cylinder of radius 6 with the x-axis as its axis and of height 2.

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Def If f is a scalar-valued function and S is a surface with parametric representation $\vec{r}(u,v)$ with $(u,v) \in R$, then the surface integral of f over S is

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_R f(u,v)\, |\vec{r}_u \times \vec{r}_v|\, dA}$$

☆ See last page

Note: $\vec{r}_u \times \vec{r}_v$ is normal to the tangent plane of S.
The proof of this definition using Riemann sums is in the book starting on page 1149.

When $f(x,y,z) = 1$, $\iint_S dS = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA$ gives the surface area of S above R.

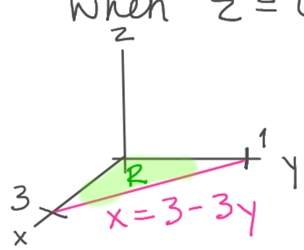
Ex.6 Find the surface area of the plane $z = 3 - x - 3y$ in the first octant using a parametric description.

$z = 3 - x - 3y$ is given, so a clear parameterization is
$\vec{r}(u,v) = \langle u, v, 3 - u - 3v\rangle$.

Now we need bounds on u and v.

In the first octant, $x \ge 0$, $y \ge 0$, $z \ge 0$.

When $z = 0$, $0 = 3 - x - 3y \Rightarrow x = 3 - 3y$, so we have



$0 \le x \le 3 - 3y$ and $0 \le y \le 1$

$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$

$0 \le u \le 3 - 3v \qquad 0 \le v \le 1$

The surface area is $\iint_S dS = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA$

$\vec{r}_u = \langle 1, 0, -1 \rangle$
$\vec{r}_v = \langle 0, 1, -3 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{vmatrix} = \langle 0-(-1), -(-3-0), 1-0 \rangle = \langle 1, 3, 1 \rangle$$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{1+9+1} = \sqrt{11}$

$$\begin{aligned} \iint_S dS &= \iint_R \sqrt{11}\, dA \\ &= \int_0^1 \int_0^{3-3v} \sqrt{11}\, du\, dv \\ &= \int_0^1 \sqrt{11}(3-3v)\, dv \\ &= \sqrt{11}\left[3v - \tfrac{3}{2}v^2\right]_0^1 \\ &= \sqrt{11}\left(3 - \tfrac{3}{2}\right) = \boxed{\tfrac{3\sqrt{11}}{2}} \end{aligned}$$

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Ex.7 Evaluate the surface integral $\iint_S y\, dS$ where S is the cylinder $x^2+y^2=9$, $0 \le z \le 3$.

Parametric description: $\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle$, $0 \le u \le 2\pi$, $0 \le v \le 3$

$\vec{r}_u = \langle -3\sin u, 3\cos u, 0 \rangle$
$\vec{r}_v = \langle 0, 0, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 3\cos u, 3\sin u, 0 \rangle$$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{9(\cos^2 u + \sin^2 u)} = 3$

$$\begin{aligned} \iint_S y\, dS &= \iint_R 3\sin u\, |\vec{r}_u \times \vec{r}_v|\, dA \\ &= \int_0^{2\pi} \int_0^3 3\sin u\,(3)\, dv\, du \\ &= 27\left[-\cos u\right]_0^{2\pi} \\ &= \boxed{0} \end{aligned}$$

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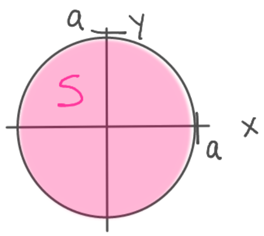
Now, we can fully explain double integrals in polar coordinates!



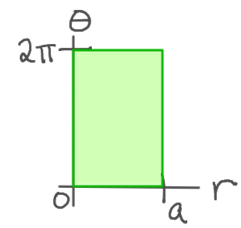
(xy-plane)

We can parameterize the circle of radius a using $\vec{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 0\rangle$

in the xy-plane, so $z=0$

with $0 \le r \le a$ and $0 \le \theta \le 2\pi$

Note that R is



because the parameters are drawn in the $r\theta$-plane.

Then $\iint_S f(x,y)\,dS = \iint_R f(r\cos\theta, r\sin\theta)\ |\vec{r}_r \times \vec{r}_\theta|\,dA$

$\vec{r}_r = \langle \cos\theta,\ \sin\theta,\ 0\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle 0, 0, r\cos^2\theta + r\sin^2\theta\rangle$$

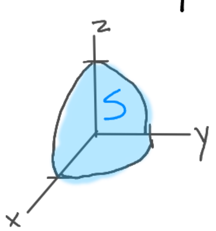
$$= \langle 0, 0, r\rangle$$

$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2} = r$

$$\iint_S f(x,y)\,dS = \iint_R f(r\cos\theta, r\sin\theta)\ |\vec{r}_r \times \vec{r}_\theta|\,dA$$

$$= \int_0^{2\pi}\int_0^a f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta$$

<u>Ex.</u> 8 Evaluate $\boxed{\iint_S f\,dS}$ → in (x,y,z) coordinates! where $\boxed{f(\rho,\varphi,\theta) = \cos\varphi}$ → in spherical and S is the part of the unit sphere in the first octant.



This part of the unit sphere is

$\{(\rho,\varphi,\theta) : \rho = 1,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2}\}$

in spherical coordinates, so

$$x = \overset{=1}{\cancel{\rho}}\sin\varphi\cos\theta$$
$$y = \overset{=1}{\cancel{\rho}}\sin\varphi\sin\theta$$
$$z = \overset{=1}{\cancel{\rho}}\cos\varphi$$

$\rho = 1$ is constant, so we use $u = \varphi$, $v = \theta$.

$$\vec{r}(u,v) = \langle \sin u \cos v,\ \sin u \sin v,\ \cos u \rangle$$

$$\vec{r}_u = \langle \cos u \cos v,\ \cos u \sin v,\ -\sin u \rangle$$
$$\vec{r}_v = \langle -\sin u \sin v,\ \sin u \cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle \sin^2 u \cos v,\ \sin^2 u \sin v,\ \underbrace{\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v}_{= \cos u \sin u \,(\cancel{\cos^2 v + \sin^2 v})} \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(\sin^2 u \cos v)^2 + (\sin^2 u \sin v)^2 + (\cos u \sin u)^2}$$
$$= \sqrt{\sin^4 u(\cancel{\cos^2 v + \sin^2 v}) + \cos^2 u \sin^2 u}$$
$$= \sqrt{\sin^2 u \,(\cancel{\sin^2 u + \cos^2 u})}$$
$$= \sin u$$

$$\iint_S f(x,y,z)\, dS = \iint_R f(\overset{\varphi}{\overset{\shortparallel}{u}}, \overset{\theta}{\overset{\shortparallel}{v}})\, |\vec{r}_u \times \vec{r}_v|\, dA$$

$f(\rho, \varphi, \theta) = \cos\varphi$

$f(x,y,z) = z$ on unit sphere since $\rho = 1$

$$= \int_0^{\pi/2}\int_0^{\pi/2} (\cos u)(\sin u)\, du\, dv$$
$$= \frac{\pi}{2}\left[\frac{1}{2}\sin^2 u\right]_0^{\pi/2}$$
$$= \boxed{\frac{\pi}{4}}$$

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Ex. 9 Evaluate $\iint_S (x^2 + y^2)\, dS$ where S is the surface with parameterization $\vec{r}(u,v) = \langle 2uv,\ u^2 - v^2,\ u^2 + v^2 \rangle$ for $0 \le u^2 + v^2 \le 1$

We have $\vec{r}(u,v)$, so $\vec{r}_u = \langle 2v,\ 2u,\ 2u \rangle$

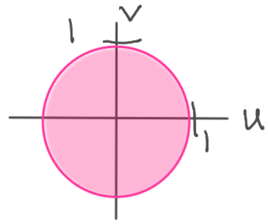
$$\vec{r}_v = \langle 2u,\ -2v,\ 2v \rangle$$
$$\vec{r}_u \times \vec{r}_v = \langle 4uv + 4uv,\ -(4v^2 - 4u^2),\ -4v^2 - 4u^2 \rangle$$
$$= \langle 8uv,\ 4u^2 - 4v^2,\ -4(u^2 + v^2) \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{64u^2v^2 + 16(u^2 - v^2)^2 + 16(u^2 + v^2)^2}$$
$$= \sqrt{16\left(4u^2v^2 + (u^4 - \cancel{2u^2v^2} + v^4) + (u^4 + \cancel{2u^2v^2} + v^4)\right)}$$
$$= 4\sqrt{2u^4 + 4u^2v^2 + 2v^4}$$
$$= 4\sqrt{2(u^2 + v^2)^2} = 4\sqrt{2}\,(u^2 + v^2)$$

$$\iint_S (x^2+y^2)\,dS = \iint_R \left((2uv)^2 + (u^2-v^2)^2\right) |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \iint_R (u^4 + 2u^2v^2 + v^4)\left(4\sqrt{2}(u^2+v^2)\right) dA$$

$$= 4\sqrt{2} \iint_R (u^2+v^2)^3\, dA$$



We can integrate this with polar coordinates in the uv-plane: $u = r\cos\theta$
$v = r\sin\theta$
$0 \le r \le 1$, $0 \le \theta \le 2\pi$

Note: We calculated $|\vec{r}_u \times \vec{r}_v|$ to switch dS in (x,y,z) to dA in (u,v). Now, to switch dA in (u,v) to polar coordinates, we use $dA = r\,dr\,d\theta$. Ignore that you previously calculated $|\vec{r}_u \times \vec{r}_v|$ and pretend as though you are encountering the integral

$\boxed{4\sqrt{2}\iint_R (u^2+v^2)\,dA, \quad R = \{(u,v) : 0 \le u^2+v^2 \le 1\}}$

with no other information. If you saw just that integral, your instinct should be to convert to polar.

$$= 4\sqrt{2} \int_0^{2\pi}\int_0^1 (r^2)^3\, r\,dr\,d\theta$$

$$= 4\sqrt{2}(2\pi)\int_0^1 r^7\,dr$$

$$= 8\sqrt{2}\,\pi \left[\tfrac{1}{8} r^8\right]_0^1$$

$$= \boxed{\sqrt{2}\,\pi} \quad \blacksquare$$

☆ Real world example of parameterizing:

The surface of the earth cannot be accurately represented in two dimensions. Instead, it's better to think of it as a parameterization where

$u =$ latitude $\quad -90° \le u \le 90°$

$v =$ longitude $\quad -180° \le v \le 180°$