§17.6 Surface Integrals (Part 2)

* A surface S is <u>explicitly defined</u> if the surface can be written as $z = g(x,y)$ over a region R in the xy-plane.

In this case, $\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$, so

$$\vec{r}_x = \langle 1, 0, g_x(x,y) \rangle \qquad \Rightarrow \vec{r}_x \times \vec{r}_y = \langle -g_x(x,y), -g_y(x,y), 1 \rangle$$
$$\vec{r}_y = \langle 0, 1, g_y(x,y) \rangle$$

$$\Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{[g_x(x,y)]^2 + [g_y(x,y)]^2 + 1}$$
$$\left(= \sqrt{g_x^2 + g_y^2 + 1} \right)$$

Then the surface integral of a (scalar-valued) function f is

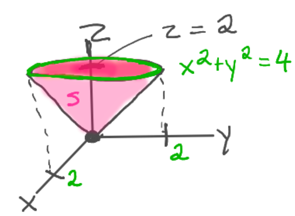
$$\boxed{\iint_S f(x,y,z)\, dS = \iint_R f(x,y,g(x,y)) \sqrt{g_x^2 + g_y^2 + 1}\, dA}$$

Be careful! g_x and g_y are scalar-valued functions while $\vec{r}_x$ and $\vec{r}_y$ are vector fields.

You do not have to memorize this formula.
It's just a specific case of $\iint_S f\, dS = \iint_R f\, |\vec{r}_u \times \vec{r}_v|\, dA$.

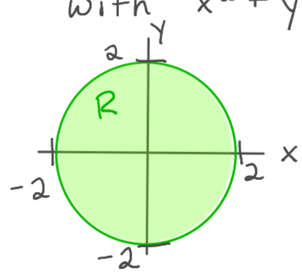
<u>Ex1</u> Find the average temperature on the cone $z^2 = x^2 + y^2$, $0 \le z \le 2$ where temperature is $T(x,y,z) = 100 - 25z$.

$$\text{Avg temp} = \frac{\text{Total Temp over } S}{\text{Surface Area of } S}$$



$$\text{Surface Area of } S = \iint_S 1\, dS$$

S is given by $\vec{r}(x,y) = \langle x, y, \sqrt{x^2 + y^2} \rangle$ with $x^2 + y^2 \le 4$



$$g(x,y) = \sqrt{x^2 + y^2}$$

$$g_x(x,y) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$g_y(x,y) = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\iint_S dS = \iint_R \sqrt{g_x^2 + g_y^2 + 1}\, dA$$

$$= \iint_R \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\, dA$$

$$= \iint_R \sqrt{\frac{x^2+y^2+(x^2+y^2)}{x^2+y^2}}\, dA$$

$$= \iint_R \sqrt{2}\, dA$$ $\rightarrow$ Since R is a circle, switch to polar to evaluate.

$$= \int_0^{2\pi}\int_0^2 \sqrt{2}\, r\,dr\,d\theta$$

$$= 2\pi \frac{\sqrt{2}}{2}(4) = 4\sqrt{2}\,\pi$$

Total Temp $= \iint_S T(x,y,z)\,dS = \iint_R (100 - 25\sqrt{x^2+y^2})\underbrace{\sqrt{2}}_{=\sqrt{g_x^2+g_y^2+1} \text{ from above}}\,dA$

factor out 25

In polar: $25\sqrt{2}\int_0^{2\pi}\int_0^2 (4-r)\,r\,dr\,d\theta$

$$= (25\sqrt{2})(2\pi)\left[2r^2 - \frac{1}{3}r^3\right]_0^2$$

$$= 50\pi\sqrt{2}\left[8 - \frac{8}{3}\right]$$

$$= \frac{800\pi\sqrt{2}}{3}$$

$$\text{Avg} = \frac{\left(\frac{800\cancel{\pi}\cancel{\sqrt{2}}}{3}\right)}{4\cancel{\pi}\cancel{\sqrt{2}}} = \boxed{\frac{200}{3}}$$ ∎

Ex.2 Evaluate $\iint_S y\,dS$ where S is the surface $z = x + y^2$ with $0 \le x \le 1$, $0 \le y \le 2$.

$g(x,y) = x + y^2 \Rightarrow g_x = 1$, $g_y = 2y$ $\Rightarrow \sqrt{g_x^2 + g_y^2 + 1} = \sqrt{2 + 4y^2}$

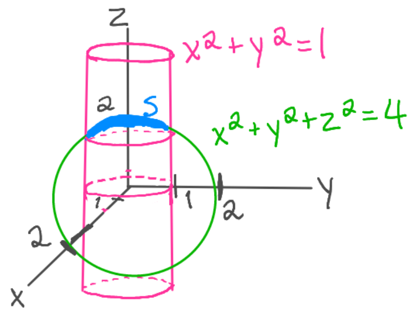
$$\iint_S y\,dS = \iint_R y\sqrt{2+4y^2}\,dA$$

$$= \int_0^1\int_0^2 y\sqrt{2+4y^2}\,dy\,dx \quad \left(= \Big(\overbrace{\int_0^1 dx}^{=1}\Big)\Big(\int_0^2 y\sqrt{2+4y^2}\,dy\Big)\right)$$

$$= \frac{1}{8}\cdot\frac{2}{3}(2+4y^2)^{3/2}\Big|_0^2$$

$$= \frac{1}{12}\left[18\sqrt{18} - 2\sqrt{2}\right] = \frac{\sqrt{2}}{6}(9(3)-1) = \boxed{\frac{13\sqrt{2}}{3}}$$ ∎

Ex.3 Evaluate $\iint_S y^2\,dS$ where S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.



$$\vec{r}(x,y) = \langle x, y, \underbrace{\sqrt{4-x^2-y^2}}_{g(x,y)} \rangle \quad \text{with } x^2+y^2 \le 1$$

$$g_x = \frac{-x}{\sqrt{4-x^2-y^2}}$$

$$g_y = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$\sqrt{g_x^2+g_y^2+1} = \sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}$$

$$= \sqrt{\frac{\cancel{x^2}+\cancel{y^2}+(4-\cancel{x^2}-\cancel{y^2})}{4-x^2-y^2}}$$

$$= \frac{2}{\sqrt{4-x^2-y^2}}$$

$$\iint_S y^2\,dS = \iint_R y^2\left(\frac{2}{\sqrt{4-x^2-y^2}}\right)dA$$

$$\text{(Polar)} \quad = \int_0^{2\pi}\int_0^1 (r\sin\theta)^2\left(\frac{2}{\sqrt{4-r^2}}\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{2r^3}{\sqrt{4-r^2}}\right)(\sin^2\theta)\,dr\,d\theta$$

$$= \left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^1 \frac{2r^{3}}{\sqrt{4-r^2}}\,dr\right) \qquad r^3 = r^2 r$$

$u = 4-r^2$
$\Rightarrow r^2 = 4-u$
$du = -2r\,dr$

$$= \left(\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta\right)\left(\int_{4=u \text{ @ } r=0}^{3=u \text{ @ } r=1} \frac{\cancel{2}(4-u)\cancel{r}}{\sqrt{u}}\left(-\frac{du}{\cancel{2r}}\right)\right) \qquad r^2 = 4-u$$

$$= \left(\frac{1}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{2\pi}\right)\left(-\int_{4=u}^{3=u}\frac{4-u}{\sqrt{u}}\,du\right) \qquad \frac{4-u}{\sqrt{u}} = 4u^{-\frac{1}{2}} - \sqrt{u}$$

$$= \left(\frac{1}{2}[2\pi]\right)\left(-\left[8u^{1/2} - \frac{2}{3}u^{3/2}\right]_4^3\right)$$

$$= -\pi\left(8\sqrt{3} - \frac{2}{\cancel{3}}(\cancel{3}\sqrt{3}) - \left(8(2) - \frac{2}{3}(8)\right)\right)$$

$$= \boxed{\left(\frac{32}{3} - 6\sqrt{3}\right)\pi}$$

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