§17.6 Surface Integrals (Part 3)

* Oriented Surfaces

If you start at a point P with normal vector pointing towards your head ($\uparrow \vec{n}$) and you can walk around a surface on a closed curve back to P and end up with the same normal vector ($\uparrow \vec{n}$ not $\downarrow \vec{n}$), then the surface is orientable.

Ex. A hemisphere is orientable

Ex. The Möbius strip is not orientable.
(We won't deal with surfaces that are not orientable, but they do exist.)

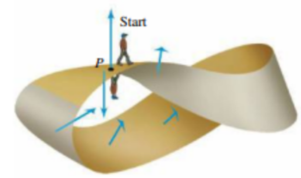
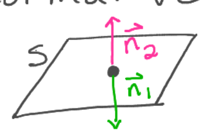


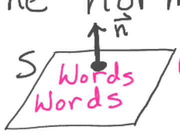
Figure 17.55

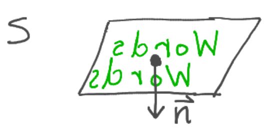
At any point on an orientable surface, there are two normal vectors $\vec{n}_1$ and $\vec{n}_2$ with $\vec{n}_2 = -\vec{n}_1$.



Choosing which vector ($\vec{n}_1$ or $\vec{n}_2$) is consistent with the orientation of S makes the orientable surface oriented.

Think about a surface with writing on one side with the normal vector on the side with writing.





Def. A surface S with (chosen) normal vector $\vec{n}$ is an oriented surface.

If S is a closed orientable surface that encloses a region (like a sphere or a cube), then, unless otherwise specified, we assume it is oriented so that the normal vectors point in the outward orientation.

For a parameterized surface given by $\vec{r}(u,v)$, the normal vector direction is $\vec{r}_u \times \vec{r}_v$ or $-(\vec{r}_u \times \vec{r}_v)$.

(The book uses $\vec{t}_u \times \vec{t}_v$ instead of $\vec{r}_u \times \vec{r}_v$.)

Def If $\vec{F} = \langle f, g, h \rangle$ is a vector field and S is an oriented surface with parameterization $\vec{r}(u,v)$ for (u,v) in R, then the <u>flux</u> or <u>surface integral of the vector field</u> is

$$\boxed{\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{n}\, dS = \iint_R \vec{F} \bullet (\vec{r}_u \times \vec{r}_v)\, dA}$$

where $\vec{n}$ is the unit normal vector whose direction is consistent with the orientation of S.

If $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$, then

$$\iint_S \underbrace{\vec{F} \bullet \vec{n}}\, dS = \iint_R \vec{F} \bullet \left(\frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \right) \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

$$= \iint_R \vec{F} \bullet (\vec{r}_u \times \vec{r}_v)\, dA$$

$\vec{F} \bullet \vec{n}$ is scalar valued function, so this is the surface integral of a function

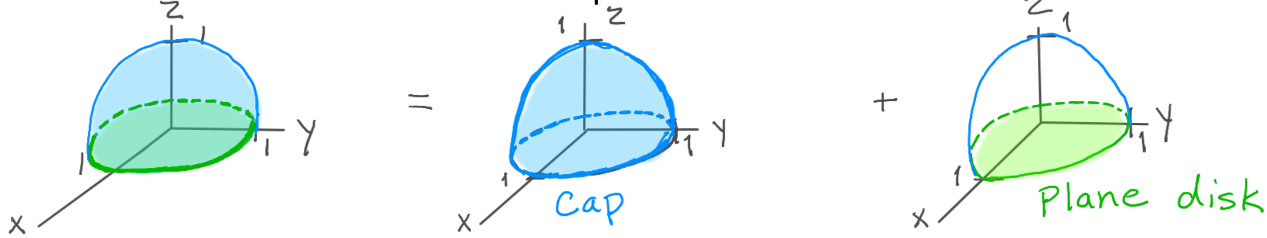
In words, the surface integral of a <u>vector field</u> over S is equal to the surface integral of <u>its normal component</u> over S.

Thinking of flux, we are only taking the part of $\vec{F}$ passing <u>through</u> S in the direction of the normal vector.

If S is explicitly defined with $z = s(x,y)$ for (x,y) in R, then $\iint_S \vec{F} \bullet \vec{n}\, dS = \iint_R (-f z_x - g z_y + h)\, dA$

(assuming $+\vec{k}$ upward orientation).

Ex1 Evaluate $\iint_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = \langle y, x, z \rangle$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



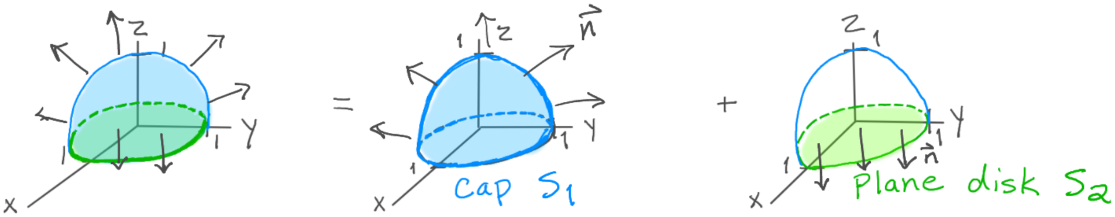
Cap: $\iint_S \vec{F} \bullet d\vec{S}$ would be computed over just the cap if the problem said, "S is the paraboloid $z = 1 - x^2 - y^2$, $0 \le z \le 1$."

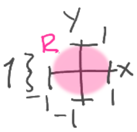
Disk: $x^2 + y^2 \le 1$ is part of the boundary for the described solid E.

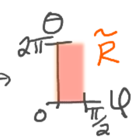
We cannot represent the cap and the disk with the same parameterization, so we will compute the surface integrals separately:

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \underset{\text{(cap)}}{\iint_{S_1}} \vec{F} \bullet n\, dS + \underset{\text{(disk)}}{\iint_{S_2}} \vec{F} \bullet \vec{n}\, dS$$

Since S is a closed surface, we may assume $\vec{n}$ is the outward normal.



Cap: $\vec{r}(x,y) = \langle x, y, 1 - x^2 - y^2 \rangle$, $R = \{(x,y) : x^2 + y^2 \le 1\}$ 

(or spherical on sphere $\rho = 1$ we have
$\vec{r}(\varphi, \theta) = \langle \sin\varphi \sin\theta, \sin\varphi \cos\theta, \cos\varphi \rangle$,
$\tilde{R} = \{(\varphi, \theta) : 0 \le \varphi \le \frac{\pi}{2}, \ 0 \le \theta \le 2\pi\} \longrightarrow$ )

With $\vec{r}(x,y)$: $\vec{r}_x = \langle 1, 0, -2x \rangle$, $\vec{r}_y = \langle 0, 1, -2y \rangle$ $\Rightarrow \vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle$

$$\iint_{S_1} \vec{F} \bullet d\vec{S} = \iint_{S_1} \vec{F} \bullet (\vec{r}_x \times \vec{r}_y)\, dS$$

$$= \iint_R \underbrace{\vec{F}(x, y, 1 - x^2 - y^2)}_{=\langle y, x, 1 - x^2 - y^2 \rangle} \bullet \langle 2x, 2y, 1 \rangle\, dA$$

$$= \iint_R (2xy + 2xy + 1 - x^2 - y^2)\, dA$$

In rectangular (x,y): $= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (4xy + 1 - x^2 - y^2)\, dx\, dy$

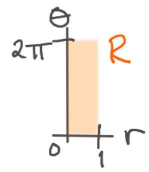
R = 
$x = -\sqrt{1-y^2}$ $x = \sqrt{1-y^2}$

$$= \int_{-1}^{1} \Big(\underbrace{2\sqrt{1-y^2}}_{\text{Trig Sub}} - \frac{2}{3} \underbrace{(1-y^2)^{3/2}}_{\text{Trig Sub}} - y^2\sqrt{1-y^2} \Big)\, dy$$

Gross!

Back to $\iint_R (4xy + 1 - x^2 - y^2)\, dA$ and use polar:

$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$
$x^2 + y^2 = r^2$



$$= \int_0^{2\pi}\int_0^1 \left(4(r\cos\theta\, r\sin\theta) + 1 - r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(4r^3\cos\theta\sin\theta + r - r^3\right) dr\,d\theta$$

$$= \int_0^{2\pi} \left[r^4\cos\theta\sin\theta + \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\cos\theta\sin\theta + \tfrac{1}{4}\right) d\theta$$

$$= \tfrac{1}{2}\sin^2\theta + \tfrac{1}{4}\theta \Big|_0^{2\pi}$$

$$= \frac{\pi}{2}$$

Disk: $\vec{r}(x,y) = \langle x, y, 0\rangle, \quad x^2 + y^2 \le 1$

$\vec{r}_x = \langle 1, 0, 0\rangle$
$\vec{r}_y = \langle 0, 1, 0\rangle$
$\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1\rangle$

Note: $\langle 0,0,1\rangle$ has upward orientation, but for this surface the normal should have downward orientation (negative z component), so $\vec{n} = -(\vec{r}_x \times \vec{r}_y) = -\langle 0, 0, 1\rangle = \langle 0, 0, -1\rangle$.

$$\iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \iint_R \langle y, x, \underset{\substack{z \text{ on disk}}}{0}\rangle \cdot \langle 0,0,-1\rangle\, dA$$

$$= \iint_R 0\, dA$$

$$= 0$$

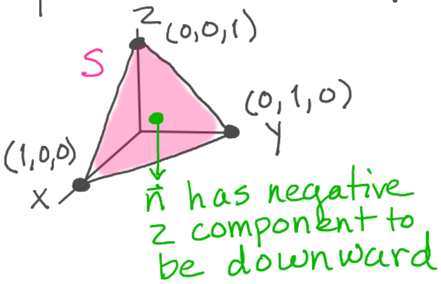
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$= \frac{\pi}{2} + 0$$

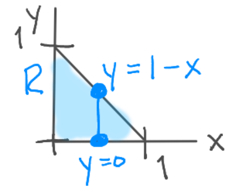
$$= \boxed{\frac{\pi}{2}}\ \blacksquare$$

Ex.2 Find $\iint_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = \langle xze^y, -xze^y, z \rangle$ and S is the part of the plane $x+y+z=1$ in the first octant with downward orientation.

First we parameterize S as $\vec{r}(x,y) = \langle x, y, 1-x-y \rangle$. We need bounds on x and y, so let's graph the plane in the first octant:



From the picture, S is the plane $z = 1-x-y$ above the triangular region when $z=0$:



$\vec{r}_x = \langle 1, 0, -1 \rangle$

$\vec{r}_y = \langle 0, 1, -1 \rangle$

$\vec{r}_x \times \vec{r}_y = \langle 1, 1, 1 \rangle$

Needs to be negative for downward orientation

$\vec{n} = -\langle 1, 1, 1 \rangle = \langle -1, -1, -1 \rangle$

$$\begin{aligned}
\vec{F} \bullet \vec{n} &= \langle xze^y, -xze^y, z \rangle \bullet \langle -1, -1, -1 \rangle \\
&= -xze^y + xze^y - z \\
&= -z
\end{aligned}$$

On S:
$$\begin{aligned}
&= -(1-x-y) \\
&= x+y-1
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \bullet \vec{n}\, dS &= \iint_R (x+y-1)\, dA \\
&= \int_0^1 \int_0^{1-x} (x+y-1)\, dy\, dx \\
&= \int_0^1 \left[xy + \tfrac{1}{2}y^2 - y \right]_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left(x(1-x) + \tfrac{1}{2}(1-x)^2 - (1-x) \right) dx \\
&= \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{6}(1-x)^3 - x + \tfrac{1}{2}x^2 \Big|_0^1 \\
&= \cancel{\tfrac{1}{2}} - \tfrac{1}{3} - 0 - \cancel{1} + \cancel{\tfrac{1}{2}} - \left(0 - 0 - \tfrac{1}{6} - 0 + 0\right) \\
&= \boxed{-\tfrac{1}{6}}
\end{aligned}$$

Ex.3 Find the flux of $\vec{F} = \langle y, x, z \rangle$ through the surface S given by $\vec{r}(u,v) = \langle u\cos v, u\sin v, u \rangle$, $0 \le u \le 2$, $\frac{\pi}{2} \le v \le \pi$ with upward orientation.

$$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$$
$$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$$
$$\Rightarrow \vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u \rangle$$

Since $0 \le u \le 2$, always positive z component, so $\vec{n} = \vec{r}_u \times \vec{r}_v$.

$$\iint_S \vec{F} \bullet \vec{n}\, d\vec{S} = \iint_R \langle u\sin v, u\cos v, u \rangle \bullet \langle -u\cos v, -u\sin v, u \rangle\, dA$$
$$= \iint_R (-2u^2\cos v\sin v + u^2)\, dA$$
$$= \int_{\pi/2}^{\pi}\int_0^2 u^2(1 - 2\cos v\sin v)\, du\, dv$$
$$= \left(\int_0^2 u^2\, du\right)\left(\int_{\pi/2}^{\pi} (1 - 2\cos v\sin v)\, dv\right)$$
$$= \left[\frac{1}{3}u^3\right]_0^2 \left[v - \sin^2 v\right]_{\pi/2}^{\pi}$$
$$= \left(\frac{8}{3}\right)\left(\pi - 0 - \left(\frac{\pi}{2} - 1\right)\right)$$
$$= \boxed{\frac{8}{3}\left(\frac{\pi}{2} + 1\right)} \blacksquare$$