§ 17.7 Stokes' Theorem (Part 1)

Recall: Circulation is the integral of a vector field along a closed curve that gives how much the field pushes the object along the curve:

$$\oint_C \vec{F} \bullet d\vec{r}$$

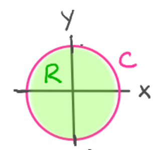
If $\vec{F} = \langle f, g \rangle$, $\oint_C \vec{F} \bullet d\vec{r} = \oint_C \langle f, g \rangle \bullet \langle dx, dy \rangle$
$= \oint_C f\,dx + g\,dy.$

If $\vec{F} = \langle f, g, h \rangle$, $\oint_C \vec{F} \bullet d\vec{r} = \oint_C \langle f, g, h \rangle \bullet \langle dx, dy, dz \rangle$
$= \oint_C f\,dx + g\,dy + h\,dz.$

We later learned when $\vec{F} = \langle f, g \rangle$, we can use Green's Theorem to compute circulation:

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R (g_x - f_y)\, dA$$

where C is the boundary curve of region R



and $g_x - f_y$ is the two dimensional curl of $\vec{F}$.

Even later, we defined the curl of a three dimensional vector field as curl $\vec{F} = \nabla \times \vec{F}$.

(Note: We can treat $\vec{F}(x,y) = \langle f, g \rangle$ as $\langle f(x,y), g(x,y), 0 \rangle$

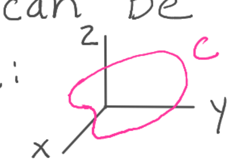
Then curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix} = \langle 0, 0, g_x - f_y \rangle$,

so the only non zero component of $\nabla \times \vec{F}$ is $g_x - f_y$ which is the two dimensional curl.)

Now that we can find the curl of a vector field $\vec{F}(x,y,z) = \langle f, g, h \rangle$, we want an analog of Green's Theorem.

However, when $\vec{F}(x,y) = \langle f, g \rangle$, we are computing $\oint_C \vec{F} \bullet d\vec{r}$ for closed curves C in the xy-plane, so C bounds a region R in the xy-plane.

If $\vec{F}(x,y,z) = \langle f, g, h \rangle$, then $\oint_C \vec{F} \bullet d\vec{r}$ can be computed over a closed curve in space:



C no longer bounds a region in a plane. Instead, it can bound a surface.
(The region R in the xy-plane is a flat surface.)

Thm Stokes' Theorem

Let S be an oriented surface in $\mathbb{R}^3$ with closed boundary curve C whose orientation is consistent with the orientation of S (use the right hand rule).

If $\vec{F} = \langle f, g, h \rangle$, then

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet \vec{n}\, dS = \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}}$$

where $\vec{n}$ is the unit normal vector to S determined by the orientation of S.

Recall: Right-Hand Rule

When you curl the fingers of your right hand in the positive direction around C, your right thumb points in the (general) direction of the vectors normal to S.

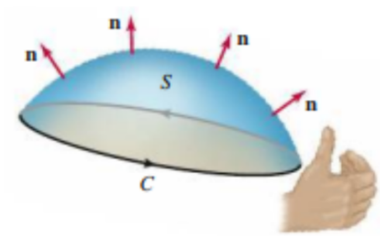


Figure 17.60

➤ The right-hand rule tells you which of two normal vectors at a point of S to use. Remember that the direction of normal vectors changes continuously on an oriented surface.

Now, we can compare Stokes' Theorem and Green's Theorem:

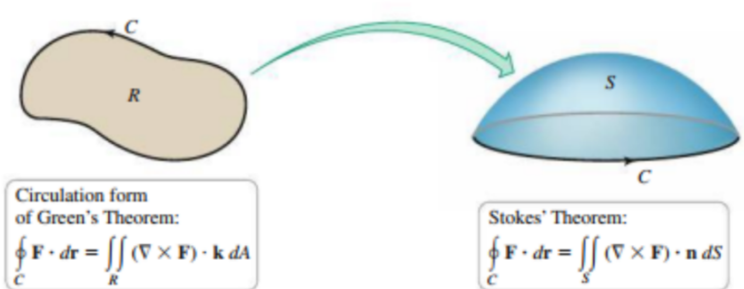


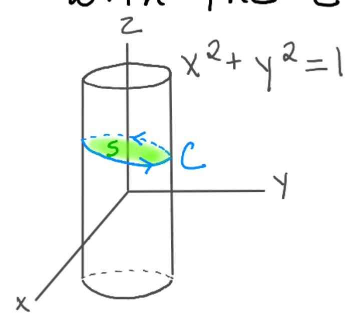
Figure 17.59

Meaning: Under proper conditions, the accumulated rotation of the vector field over the surface S (given by the normal component of the curl) equals the net circulation on the boundary of S.

Remember from §17.3 that the circulation of a conservative vector field on a closed curve is 0.

You can use Stokes' Theorem to compute line integrals $\oint_C \vec{F} \cdot d\vec{r}$ for problems where one or more terms of $\oint_C f\,dx + g\,dy + h\,dz$ cannot be integrated (ex: $\int_0^1 e^{t^2} dt$). It can also be used to compute the surface integral of the curl of a vector field: $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$.
However, we cannot use Stokes' Theorem to compute $\iint_S \vec{F} \cdot d\vec{S}$ unless we know $\vec{F} = \text{curl}\,\vec{G}$ for some vector field $\vec{G}$.

Ex.1 Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y+z=2$ with the cylinder $x^2+y^2=1$.



The curve C is the closed boundary curve of surface S.
We can write a parametric description of C where x and y line on the circle in the xy-plane $x^2+y^2=1$ and z is given by the plane $z=2-y$.
Then $\vec{r}(t) = \langle \underbrace{\cos t, \sin t}_{x^2+y^2=1}, \underbrace{2-\sin t}_{z=2-y} \rangle$, $0 \le t \le 2\pi$

Without Stokes' Theorem, we can compute as follows:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \langle -y^2, x, z^2 \rangle \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin^2 t, \cos t, (2-\sin t)^2 \rangle \cdot \langle -\sin t, \cos t, -\cos t \rangle dt$$

$$= \int_0^{2\pi} \left(\sin^3 t + \cos^2 t - \cos t\,(2-\sin t)^2\right) dt$$

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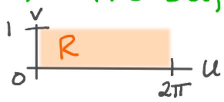
$$= \pi$$

To use Stokes', we need to parameterize the surface S. We can think about cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, but $z = 2 - y = 2 - r\sin\theta$ in cylindrical coordinates. To avoid confusion, use $v = r$ and $u = \theta$.

(or $v = \theta$ and $u = r$)

$$\vec{r}(u,v) = \langle v\cos u,\ v\sin u,\ 2 - v\sin u\rangle,\quad 0 \le u \le 2\pi,\quad 0 \le v \le 1$$

$\Rightarrow$ 

We also need curl $\vec{F} = \nabla \times \vec{F}$. We need to compute $\nabla \times \vec{F}$ in terms of (x,y,z) because $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$. We cannot easily compute $\nabla \times \vec{F}$ with $\vec{F}(\vec{r}(u,v))$.

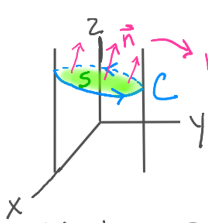
$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0-0, -(0-0), 1-(-2y)\rangle = \langle 0, 0, 1+2y\rangle$$

$$= \langle 0, 0, 1 + 2v\sin u\rangle \quad (\text{on } S)$$

We also need to find the normal vector to S consistent with the positive (counterclockwise) orientation of C:

$$\vec{r}_u = \langle -v\sin u,\ v\cos u,\ -v\cos u\rangle$$
$$\vec{r}_v = \langle \cos u,\ \sin u,\ -\sin u\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -v\cos u\sin u + v\cos u\sin u,\ -(v\sin^2 u + v\cos^2 u),\ -v\sin^2 u - v\cos^2 u\rangle$$
$$= \langle 0, -v, -v\rangle$$



normals consistent with orientation of C, so we need normals with positive z components

$$\vec{n} = -(\vec{r}_u \times \vec{r}_v) = -\langle 0, -v, -v\rangle = \langle 0, v, v\rangle$$

Using Stokes': $\oint_C \vec{F} \bullet d\vec{r} = \iint_S \text{curl } \vec{F} \bullet d\vec{S} \quad \left(= \iint_S (\nabla \times \vec{F}) \bullet \vec{n}\, dS\right)$

$$= \iint_R \langle 0, 0, 1 + 2v\sin u\rangle \bullet \langle 0, v, v\rangle\, dA$$

$$= \int_0^1 \int_0^{2\pi} (v + 2v^2 \sin u)\, du\, dv$$

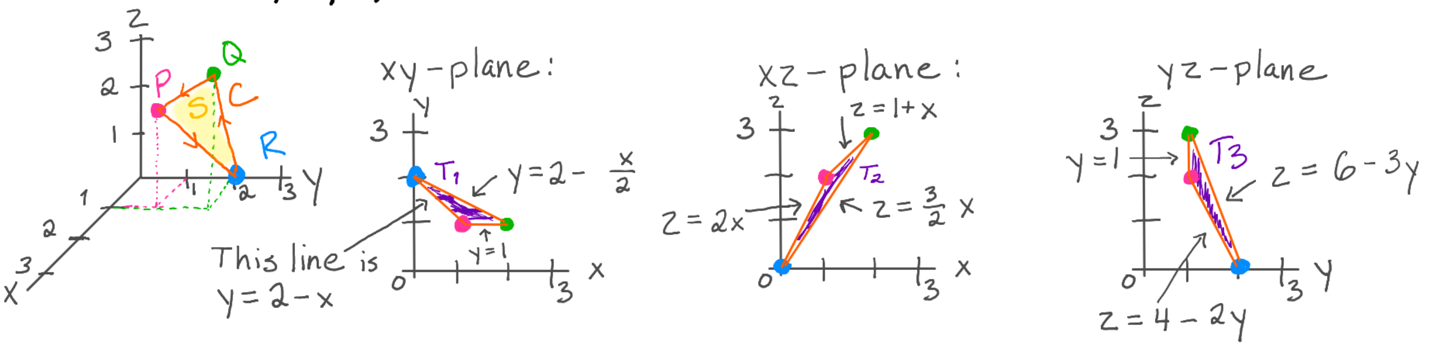
If parameterized with (θ, r) instead of (u,v), we would have (θ, r) over the same R, so we would <u>not</u> use $dA = r\,dr\,d\theta$. As we did with (u,v), we would use $dA = dr\,d\theta$ <u>or</u> $dA = d\theta\, dr$.)

$$= \int_0^1 \left[uv - 2v^2\cos u\right]_{u=0}^{u=2\pi} dv$$

$$= \int_0^1 \left(2\pi v - \cancel{2v^2} - (0 - \cancel{2v^2})\right) dv$$

$$= \pi v^2 \Big|_0^1 = \boxed{\pi} \quad \blacksquare$$

Ex. 2 Find $\oint_C \vec{F} \bullet d\vec{r}$ when $\vec{F} = \langle z^2, y^2, xy \rangle$ and C is the counterclockwise triangle with vertices $P(1,1,2)$, $Q(2,1,3)$, and $R(0,2,0)$.



S is part of the plane containing the non collinear points. The normal to the plane can be found using $\overrightarrow{PQ} \times \overrightarrow{PR}$: $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ -1 & 1 & -2 \end{vmatrix} = \langle 0-1, -(-2+1), 1-0 \rangle = \langle -1, 1, 1 \rangle$

An equation of the plane is $-1(x-0) + 1(y-2) + 1(z-0) = 0$

$$\Rightarrow -x + y + z = 2$$

We can write S in 3 ways:

① The part of $z = 2 + x - y$ over the region T_1

② The part of $y = 2 + x - z$ over the region T_2

③ The part of $x = y + z - 2$ over the region T_3

Using ①: $\vec{r}(x,y) = \langle x, y, 2 + x - y \rangle$ with $2 - y \le x \le 4 - 2y$, $1 \le y \le 2$.

We know that $\langle -1, 1, 1 \rangle$ is normal to the plane and is consistent with the orientation of C.

To use Stokes', we need to compute

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix} = \langle x - 0, -(y - 2z), (0-0) \rangle = \langle x, 2z - y, 0 \rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S \operatorname{curl}\vec{F} \bullet d\vec{S}$$

$$= \iint_R \langle x, 2z - y, 0 \rangle \bullet \langle -1, 1, 1 \rangle \, dA$$

$$= \int_1^2 \int_{2-y}^{4-2y} (-x + 2\underbrace{z}_{=2+x-y} - y) \, dx \, dy$$

$$= \int_1^2 \int_{2-y}^{4-2y} (4 + x - 3y)\, dx\, dy$$
$$= \int_1^2 \left[4x + \tfrac{1}{2}x^2 - 3xy\right]_{x=2-y}^{x=4-2y} dy$$
$$= \int_1^2 \Big(16 - 8y + \tfrac{1}{2}(4-2y)^2 - 3y(4-2y) - \big(8 - 4y + \tfrac{1}{2}(2-y)^2 - 3y(2-y)\big)\Big)\, dy$$
$$= \int_1^2 \left(8 - 10y + 3y^2 + \tfrac{1}{2}(4-2y)^2 - \tfrac{1}{2}(2-y)^2\right) dy$$
$$= \left[8y - 5y^2 + y^3 + \tfrac{1}{6} \cdot \tfrac{-1}{2}(4-2y)^3 - \tfrac{1}{6}\cdot(-1)(2-y)^3\right]_1^2$$
$$= \left[16 - 20 \cancel{+8} - \cancel{\tfrac{1}{12}(0)} + \cancel{\tfrac{1}{6}(0)} - \left(\cancel{8} - 5 + 1 - \tfrac{8}{12} + \tfrac{1}{6}\right)\right]$$
$$= \frac{2}{3} - \frac{1}{6} = \frac{3}{6} = \boxed{\frac{1}{2}}$$

Not using Stokes', $C = \overrightarrow{PQ} \to \overrightarrow{QR} \to \overrightarrow{RP}$ which are line segments that we can parameterize as follows:

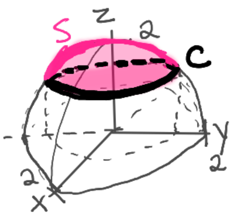
$\overrightarrow{PQ}$: $\vec{r}_1(t) = (1-t)\langle 1, 1, 2\rangle + t\langle 2, 1, 3\rangle = \langle t+1, 1, t+2\rangle$, $0 \le t \le 1$

$\overrightarrow{QR}$: $\vec{r}_2(t) = (1-t)\langle 2,1,3\rangle + t\langle 0,2,0\rangle = \langle 2-2t, t+1, 3-3t\rangle$, $0 \le t \le 1$

$\overrightarrow{RP}$: $\vec{r}_3(t) = (1-t)\langle 0,2,0\rangle + t\langle 1,1,2\rangle = \langle t, 2-t, 2t\rangle$, $0 \le t \le 1$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_{\overrightarrow{PQ}} \vec{F} \bullet d\vec{r}_1 + \int_{\overrightarrow{QR}} \vec{F} \bullet d\vec{r}_2 + \int_{\overrightarrow{RP}} \vec{F} \bullet d\vec{r}_3$$
$$= \int_0^1 \langle (t+2)^2, 1^2, (t+1)(1)\rangle \bullet \langle 1, 0, 1\rangle\, dt$$
$$+ \int_0^1 \langle (3-3t)^2, (t+1)^2, (2-2t)(t+1)\rangle \bullet \langle -2, 1, -3\rangle\, dt$$
$$+ \int_0^1 \langle (2t)^2, (2-t)^2, t(2-t)\rangle \bullet \langle 1, -1, 2\rangle\, dt$$
$$= \int_0^1 \Big((t+2)^2 + (t+1) - 2(3-3t)^2 + (t+1)^2 - 3(2-2t)(t+1) + 4t^2 - (2-t)^2 + 2t(2-t)\Big)\, dt$$
$$= \int_0^1 (-9t^2 + 51t - 22)\, dt$$
$$= -3t^3 + \frac{51}{2}t^2 - 22t \,\Big|_0^1$$
$$= -3 + \frac{51}{2} - 22$$
$$= \frac{-6 + 51 - 44}{2} = \boxed{\frac{1}{2}} \quad \blacksquare$$

Ex.3 Use Stokes' Theorem to evaluate $\iint_S \text{curl}\,\vec{F} \bullet d\vec{S}$, where $\vec{F} = \langle yz, xz, xy \rangle$ and S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane



C is the curve $x^2+y^2=1$ in the plane $z=\sqrt{3}$, so $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$, $0 \le t \le 2\pi$

By Stokes' Theorem:

$$\iint_S \text{curl}\,\vec{F} \bullet d\vec{S} = \oint_C \vec{F} \bullet d\vec{r}$$

$$= \int_0^{2\pi} \langle \sqrt{3}\sin t, \sqrt{3}\cos t, \cos t \sin t \rangle \bullet \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\sin^2 t + \sqrt{3}\cos^2 t + 0)\, dt$$

$$= \sqrt{3} \int_0^{2\pi} (\cos^2 t - \sin^2 t)\, dt$$

$$= \sqrt{3} \int_0^{2\pi} \cos(2t)\, dt$$

$$= \frac{\sqrt{3}}{2} \sin(2t) \Big|_0^{2\pi}$$

$$= \boxed{0}$$

Without Stokes', we parameterize S with $\vec{r}(u,v) = \langle 2\sin v \cos u, 2\sin v \sin u, 2\cos v \rangle$ for $0 \le u \le 2\pi$, $0 \le v \le \frac{\pi}{6}$

To find the bounds on v, recall spherical coordinates:

$$z = 2\cos v$$
$$\sqrt{3} = 2\cos v$$
$$\frac{\sqrt{3}}{2} = \cos v \quad \Rightarrow \quad v = \frac{\pi}{6}$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \langle x-x, -(y-y), z-z \rangle = \vec{0}$$

$$\iint_S \text{curl}\,\vec{F} \bullet d\vec{S} = \iint_R 0 \, dA = \boxed{0}$$

OR: Notice that $\vec{F}$ is conservative with $\vec{F} = \nabla \varphi$ for $\varphi(x,y,z) = xyz$. Since C is a closed curve, $\iint_S \text{curl}\,\vec{F} \bullet d\vec{S} = \oint_C \vec{F} \bullet d\vec{r} = 0$. ■