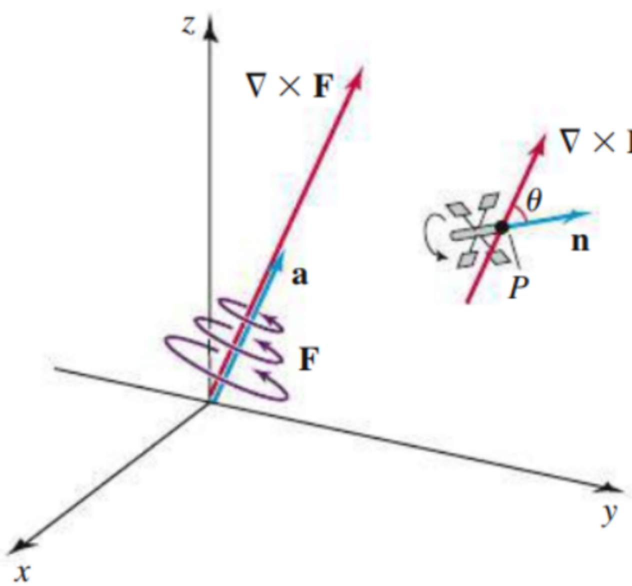
§17.7 Stokes' Theorem (Part 2)

Recall: Stokes' Theorem $\oint_C \vec{F} \bullet \vec{r}'(t)\,dt = \iint_S \operatorname{curl} \vec{F} \bullet \vec{n}\,dS$



Paddle wheel at P with axis **n** measures rotation about **n**. Rotation is a maximum when $\nabla \times \mathbf{F}$ is parallel to **n**.

Figure 17.42

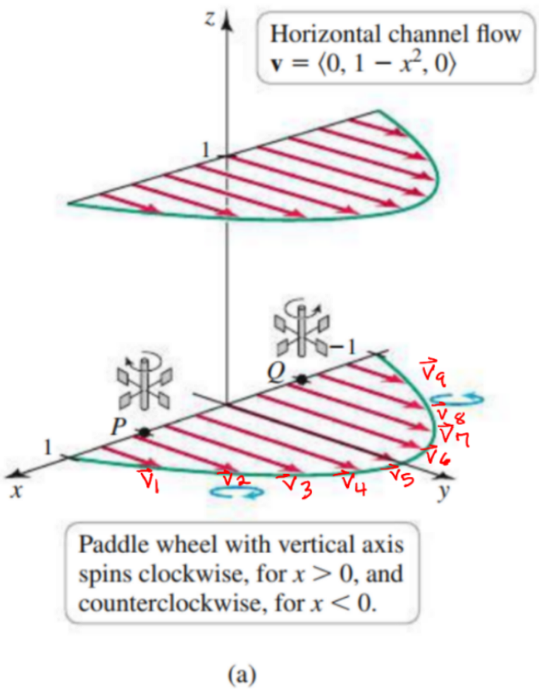
Looking at the left side of Fig 17.42. If we placed a paddle wheel at the origin tilted so that $\nabla \times \vec{F}$ were its axis, then rotation (angular speed) due to field $\vec{F}$ would be maximized.

If instead we look at the right side where we have a paddle wheel with $\vec{n}$ as its axis, then $\vec{F}$ will still cause some rotation of the paddle wheel, but its magnitude will be less. The curl of $\vec{F}$ tells us which vector to use as the axis for a paddle wheel to maximize rotation due to $\vec{F}$.

Note: Rotation is maximized when $\nabla \times \vec{F}$ is parallel to $\vec{n}$.

Thm If $\vec{F}$ is a conservative vector field, then $\operatorname{curl} \vec{F} = 0$.

(For most regions D that we encounter in this course, the converse is true. Meaning, if $\operatorname{curl} \vec{F} = 0$ on D, then $\vec{F}$ is conservative on D.)



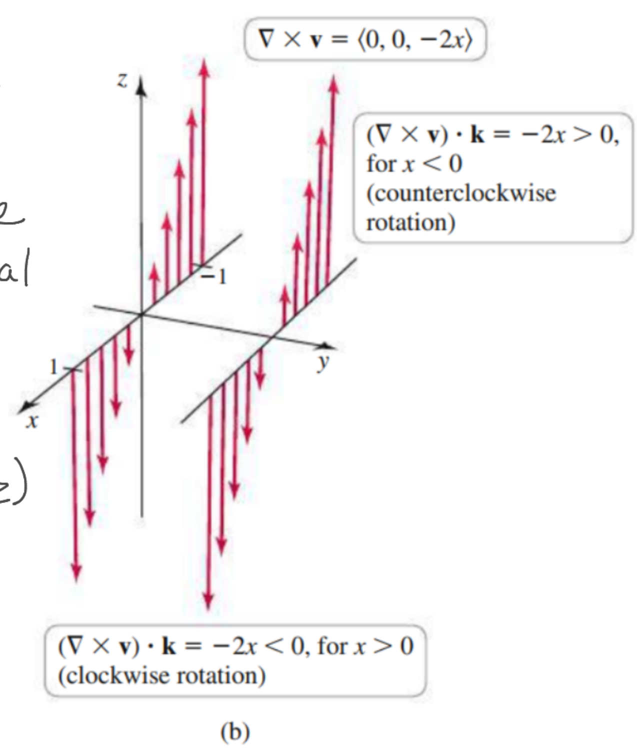
(a)

Figure 17.65

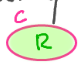
The vector field is $\vec{v} = \langle 0, 1-x^2, 0\rangle$, so at any point, $\vec{v}$ is parallel to the y-axis.
If we place a paddle wheel at point P, its axis should be parallel to the z-axis in order for $\vec{v}$ to cause the wheel to spin. If the axis is parallel to the x- or y-axes, then $\vec{v}$ will not push the paddles and cause the wheel to spin.

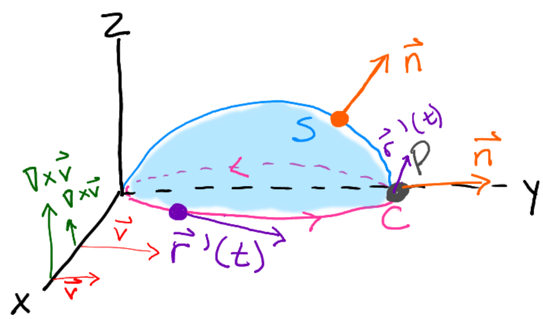
The next thing to note is that $|\vec{v}_3| > |\vec{v}_2|$, so the paddle wheel will spin clockwise at P.
If the wheel is placed at Q, the axis should still be parallel to the z-axis. Since $|\vec{v}_7| > |\vec{v}_8|$, the paddle wheel will have (net) counterclockwise rotation.

Now, we look at (b) which is a graph of curl $\vec{F} = \langle 0, 0, -2x\rangle$.
For any surface parallel to the xy-plane, $\vec{k} = \langle 0,0,1\rangle$ is a normal vector. Since $(\nabla \times \vec{v}) \bullet \vec{k} = -2x$, the magnitude of the rotation of a flywheel at a point (x,y,z) is dependent only on x (provided its axis is parallel to the z-axis).



(b)

Returning to Stokes' Theorem, consider surface S with boundary curve C:
(If , R is not part of S.)



We can parameterize C with $\vec{r}(t)$ and then $\vec{r}'(t)$ is tangent to C at any point on C.
We also have the normal vector $\vec{n}$ to S given by the parameterization of S and consistent with the orientation of C.

If we use the field $\vec{v} = \langle 0, 1-x^2, 0 \rangle$ from before, then at point P on C, $\vec{r}'(t)$ is orthogonal to $\vec{v}$, so $\vec{v} \bullet \vec{r}'(t) = 0$.

However, at point P on S, $\vec{n}$ is parallel to $\vec{v}$, so $\vec{v} \bullet \vec{n} \neq 0$. We do have at P on S that $\operatorname{curl} \vec{v}$ is orthogonal to $\vec{n}$.

This is a small example to show why Stokes' Theorem makes sense.

$$\boxed{\oint_C \vec{v} \bullet \vec{r}'(t)\, dt = \iint_S (\nabla \times \vec{v}) \bullet \vec{n}\, dS}$$

Semantics:

① A problem states: Use Stokes' Theorem to find the circulation of $\vec{F} = \nabla\varphi$ around a simple, closed curve C.

How to show your work: Circulation $= \oint_C \vec{F} \cdot dr$

$= \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$

by Stokes' Theorem. $\vec{F} = \nabla\varphi$ is conservative, and curl $\vec{F} = 0$ for a conservative vector field. This implies that the circulation is 0.

② A problem states: Find the circulation of $\vec{F} = \nabla\varphi$ around a simple, closed curve C.

The work for ① would be correct <u>OR</u>...

Different Work: Circulation $= \oint_C \vec{F} \cdot d\vec{r}$

$\vec{F} = \nabla\varphi$ is conservative, and C is a closed curve, so by the Fundamental Theorem for Line Integrals, the circulation is 0.

Both examples of showing work state that the field is conservative (for a specific $\vec{F}$, you should <u>show</u> that $\vec{F}$ is conservative) and state the theorem that allows you to determine information about the integral.

Ex.1 Consider the tilted disk C:

$$\vec{r}(t) = \langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle \text{ for } 0 \le t \le 2\pi$$

φ is a fixed angle with $0 \le \varphi \le \frac{\pi}{2}$.

(a) Use Stokes' Theorem to find the circulation on C of $\vec{F} = \langle -y, -z, x \rangle$ as a function of φ.

(b) For what value of φ is the circulation a maximum?

(a) We can parameterize the surface enclosed by C by introducing another parameter for the radius. Take $u = t$ and $v =$ radius:

$$\vec{r}(u,v) = \langle v\cos\varphi \cos u, v \sin u, v \sin\varphi \cos u \rangle$$

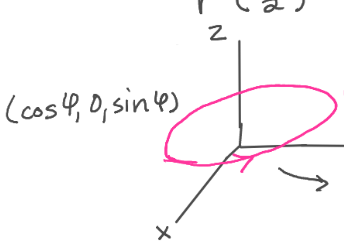
with $0 \le v \le 1$ and $0 \le u \le 2\pi$.

$$\vec{r}_u = \langle -v\cos\varphi \sin u,\ v\cos u,\ -v \sin\varphi \sin u \rangle$$
$$\vec{r}_v = \langle \cos\varphi \cos u,\ \sin u,\ \sin\varphi \cos u \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle \underline{v \sin\varphi} \cos^2 u + \underline{v \sin\varphi} \sin^2 u,$$
$$-(-v\cos\varphi \sin\varphi \sin u \cos u + v\cos\varphi \sin\varphi \cos u \sin u),$$
$$\underline{-v\cos\varphi} \sin^2 u - \underline{v\cos\varphi} \sin^2 u \rangle$$

$$= \langle v \sin\varphi,\ 0,\ -v\cos\varphi \rangle$$

Note: $\vec{r}(0) = \langle \cos\varphi, 0, \sin\varphi \rangle$ with $\sin\varphi, \cos\varphi \ge 0$
$\vec{r}(\frac{\pi}{2}) = \langle 0, 1, 0 \rangle$ for $0 \le \varphi \le \frac{\pi}{2}$



$\vec{r}(\pi) = \langle -\cos\varphi, 0, -\sin\varphi \rangle$

$\rightarrow$ Need $\vec{n}$ to have positive z component:

$$\vec{n} = \langle -v\sin\varphi, 0, v\cos\varphi \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS$$
$$= \int_0^{2\pi} \int_0^1 \langle 1, -1, 1 \rangle \cdot \langle -v\sin\varphi, 0, v\cos\varphi \rangle \, dv\, \underline{du}$$
$$= 2\pi \int_0^1 (-v\sin\varphi + v\cos\varphi)\, dv$$

$$= \cancel{2}\pi \left[\frac{1}{\cancel{2}} v^2 (-\sin\varphi + \cos\varphi) \right]_0^1$$

$$= \boxed{\pi (\cos\varphi - \sin\varphi)}$$

(b) Circulation $= \underbrace{g(\varphi)}_{\text{function of 1 variable, so}} = \pi(\cos\varphi - \sin\varphi)$

abs max prob from calc 1

$$g'(\varphi) = \pi(-\sin\varphi - \cos\varphi)$$

$$0 = \pi(-\sin\varphi - \cos\varphi)$$

$$\cos\varphi = -\sin\varphi$$

$\Rightarrow \varphi = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$ which are not in $0 \le \varphi \le \frac{\pi}{2}$

Check endpoints: $\varphi = 0 \Rightarrow g(0) = \pi(1-0) = \pi$

$\varphi = \frac{\pi}{2} \Rightarrow g(\frac{\pi}{2}) = \pi(0-1) = -\pi$

Circulation has maximum value of π when $\boxed{\varphi = 0}$ ∎