§17.8 Divergence Theorem (Part 1)

Recall: Flux is the amount of a substance that flows across or through a surface.

In 2D: $\int_C \vec{F}\cdot\vec{n}\,ds = \int_C f\,dy - g\,dx$

$\vec{F}=\langle f,g\rangle$

For closed C... $= \iint_R \underbrace{(f_x + g_y)}_{\text{the two-dimensional divergence of } \vec{F}} dA$ by Green's Thm

In 3D: $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$ if consistent with orientation given by $\vec{n}$.

$\vec{F}=\langle f,g,h\rangle$

In §17.7, we saw that Stokes' Theorem gives us a way to compute $\oint_C \vec{F}\cdot d\vec{r}$ for $\vec{F}=\langle f,g,h\rangle$ using the curl of $\vec{F}$.

The Divergence Theorem will give us a way to compute $\iint_S \vec{F}\cdot\vec{n}\,dS$ using the divergence of $\vec{F}$.

Thm Divergence Theorem (net outward flux)

If $\vec{F}=\langle f,g,h\rangle$ and D is a region in $\mathbb{R}^3$ enclosed by an oriented surface S with outward unit normal $\vec{n}$,

then $$\boxed{\underbrace{\iint_S \vec{F}\cdot\vec{n}\,dS}_{\substack{\text{measures how much of } \vec{F} \\ \text{passes through the surface } S}} = \underbrace{\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D \operatorname{div}\vec{F}\,dV}_{\substack{\text{measures the net contraction or expansion} \\ \text{of } \vec{F} \text{ inside the surface } S}}}$$

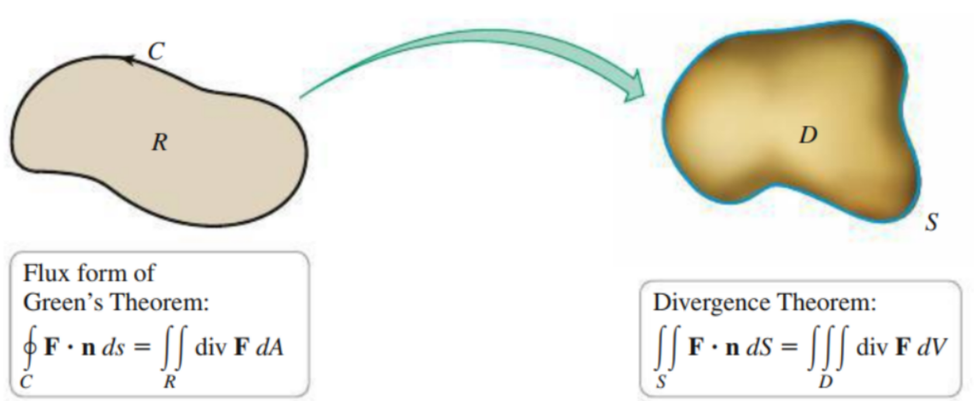
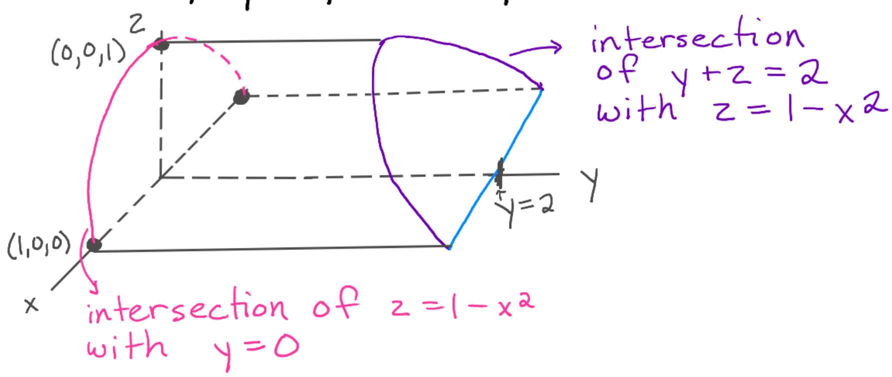


Figure 17.68

Ex. 1 Find the net outward flux of $\vec{F}(x,y,z) = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$ across the surface S of the solid region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.



Solid region is bounded by

$0 \le y \le 2 - z$

$0 \le z \le 1 - x^2$

$-1 \le x \le 1$

Net outward flux $= \iint_S \vec{F} \bullet \vec{n}\, dS$

(Without Divergence Thm) $= \iint_{\text{Bottom}} + \iint_{\text{Left}} + \iint_{\text{Right}} + \iint_{\text{Top (curved part)}}$

The easiest surfaces to evaluate are on the left (part of the plane $y=0$ where $z = 1-x^2$ and $-1 \le x \le 1$) and the bottom (part of $z=0$ where $-1 \le x \le 1$, $0 \le y \le 2$).

Bottom: $\vec{r}(x,y) = \langle x, y, 0 \rangle$ $\quad \vec{r}_x = \langle 1, 0, 0 \rangle$ $\quad \vec{r}_y = \langle 0, 1, 0 \rangle$ $\quad \vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle$

outward flux $\Rightarrow n = -(\vec{r}_x \times \vec{r}_y) = \langle 0, 0, -1 \rangle$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \int_{-1}^{1} \int_0^2 \langle xy,\ y^2 + 1,\ \sin(xy) \rangle \bullet \langle 0, 0, -1 \rangle\, dy\, dx$$

$$= \int_{-1}^{1} \int_0^2 -\sin(xy)\, dy\, dx$$

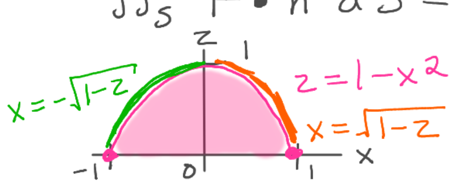
$$= \int_{-1}^{1} \left[\frac{\cos(xy)}{x} \right]_{y=0}^{y=2} dx$$

Improper integral! Not defined for $x = 0$. Would need to cancel with another term somehow $\leftarrow$ $= \int_{-1}^{1} \left(\frac{\cos(2x)}{x} - \frac{1}{x} \right) dx$

Left: $\vec{r}(x,z) = \langle x, 0, z \rangle$ $\quad \vec{r}_x \times \vec{r}_z = \langle 0, -1, 0 \rangle = \vec{n}$ (outward)

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iint_R e^{xz^2}\, dA = \int_{-1}^{1} \int_0^{1-x^2} e^{xz^2}\, dz\, dx$$

↑ can't integrate dz first



$$= \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} e^{xz^2}\, dx\, dz$$

$$= \int_0^1 \left[\frac{e^{xz^2}}{z^2} \right]_{-\sqrt{1-z}=x}^{\sqrt{1-z}=x} dz$$

Improper and hard to integrate. $\leftarrow$ $= \int_0^1 \left[\frac{e^{z^2\sqrt{1-z}}}{z^2} - \frac{e^{-z^2\sqrt{1-z}}}{z^2} \right] dz$

Instead, use Divergence Theorem!

$$\operatorname{div} \vec{F} = \nabla \bullet \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$$

$$= \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \bullet \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$$

$$= y + 2y + 0$$

$$= 3y$$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$= \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx$$

$$= \int_{-1}^1 \int_0^{1-x^2} \tfrac{3}{2}(2-z)^2\, dz\, dx$$

$$= \int_{-1}^1 -\tfrac{1}{2} \Big(\underbrace{(x^2+1)^3}_{x^6 + 3x^4 + 3x^2 + 1} - 8 \Big)\, dx$$

$$= -\tfrac{1}{2} \left[\tfrac{1}{7} x^7 + \tfrac{3}{5} x^5 + x^3 - 7x \right]_{-1}^1$$

$$= -\left(\tfrac{1}{7} + \tfrac{3}{5} + 1 - 7 \right)$$

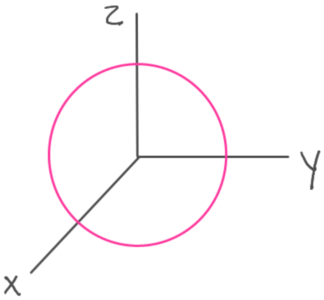
$$= -\frac{5 + 21 - 6(35)}{35} = \boxed{\frac{184}{35}} \quad \blacksquare$$

Ex.2 Find the net outward flux of $\vec{F} = \langle x^3 + y^3,\ y^3 + z^3,\ z^3 + x^3 \rangle$ across the sphere of radius 2 centered at the origin.

Want to compute $\iint_S \vec{F} \bullet \vec{n}\, dS$.

Divergence Thm says $= \iiint_E \operatorname{div} \vec{F}\, dV$.

Without Divergence Thm ...



We can parameterize S by using our knowledge of spherical coordinates:

$$\rho = 2 \qquad x = \rho \sin\varphi \cos\theta$$
$$0 \le \varphi \le \pi \qquad y = \rho \sin\varphi \sin\theta$$
$$0 \le \theta \le 2\pi \qquad z = \rho\cos\varphi$$

$$\Rightarrow \vec{r}(u,v) = \langle 2\sin u\cos v,\ 2\sin u \sin v,\ 2\cos u\rangle$$
$$0 \le u \le \pi \quad \text{and} \quad 0 \le v \le 2\pi$$

$$\Rightarrow \vec{r}_u = \langle 2\cos u\cos v,\ 2\cos u\sin v,\ -2\sin u\rangle$$
$$\vec{r}_v = \langle -2\sin u\sin v,\ 2\sin u\cos v,\ 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\cos u\sin u\rangle$$

$(= \langle (2\sin u)\,x,\ (2\sin u)\,y,\ (2\sin u)\,z\rangle$ which is in the outward direction$)$

$$\iint_S \vec{F}\bullet\vec{n}\,dS$$

$$= \iint_R \langle 8\sin^3 u(\cos^3 v + \sin^3 v),\ 8(\sin^3 u\sin^3 v + \cos^3 u),\ 8(\cos^3 u + \sin^3 u\cos^3 v)\rangle$$
$$\bullet \langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\cos u\sin u\rangle\, dA$$

$$= 32\iint_R \big(\sin^5 u\cos v(\cos^3 v + \sin^3 v) + \sin^2 u\sin v(\sin^3 u\sin^3 v + \cos^3 u)$$
$$+ \cos u\sin u(\cos^3 u + \sin^3 u\cos^3 v)\big)\, dA$$

Really messy!

Using Divergence Theorem...

We restarted the problem, so prior to this, we had not parameterized anything.

$$\operatorname{div}\vec{F} = \nabla\bullet\langle x^3+y^3,\ y^3+z^3,\ z^3+x^3\rangle$$
$$= 3x^2 + 3y^2 + 3z^2$$

$$\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E 3(x^2+y^2+z^2)\,\underline{dV}$$

Use spherical!

solid ball of radius 2 centered at origin

Because the inside is $\underline{\text{solid}}$ $0 \le \rho \le 2$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3\rho^2\,\underline{(\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta)}$$

$$= \left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\pi}\sin\varphi\, d\varphi\right]\left[\int_0^{2} 3\rho^4\, d\rho\right]$$

$$= (2\pi)(2)\left(\tfrac{3}{5}\cdot(2)^5\right) = \boxed{\frac{384\pi}{5}}$$

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