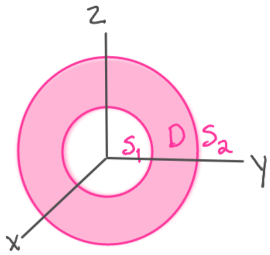
§17.8 Divergence Theorem (Part 2)

Thm Divergence Theorem for Hollow Regions

Suppose D is a region bounded by two oriented surfaces S_1 and S_2 where S_1 lies within S_2. If S is the entire boundary $(S = S_1 \cup S_2)$, then

$$\boxed{\begin{aligned}\iiint_D \nabla \cdot \vec{F}\, dV &= \iint_S \vec{F} \cdot \vec{n}\, dS \\ &= \iint_{S_2} \vec{F} \cdot \vec{n}_2\, dS - \iint_{S_1} \vec{F} \cdot \vec{n}_1\, dS\end{aligned}}$$

Ex. 1 Compute the net outward flux of $\vec{F} = \langle 3y, \frac{2}{7}x^2, 2z \rangle$ where D is the region between the spheres of radius 1 and 2 centered at the origin.



To set up the surface integrals...

S_1: $\vec{r}_1(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$ $\quad 0 \le u \le \pi$, $0 \le v \le 2\pi$

$\vec{r}_{1,u} = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$

$\vec{r}_{1,v} = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$

$\vec{n}_1 = \langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u \rangle$

S_2: $\vec{r}_2(u,v) = \langle 2\sin u \cos v, 2\sin u \sin v, 2\cos u \rangle$ $\quad 0 \le u \le \pi$, $0 \le v \le 2\pi$

$\vec{r}_{2,u} = \langle 2\cos u \cos v, 2\cos u \sin v, -2\sin u \rangle$

$\vec{r}_{2,v} = \langle -2\sin u \sin v, 2\sin u \cos v, 0 \rangle$

$\vec{n}_2 = \langle 4\sin^2 u \cos v, 4\sin^2 u \sin v, 4\cos u \sin u \rangle$

net outward flux $= \iint_S \vec{F} \cdot \vec{n}\, dS$

$= \iint_{S_2} \vec{F}(\vec{r}_2(u,v)) \cdot \vec{n}_2\, dS - \iint_{S_1} \vec{F}(\vec{r}_1(u,v)) \cdot \vec{n}_1\, dS$

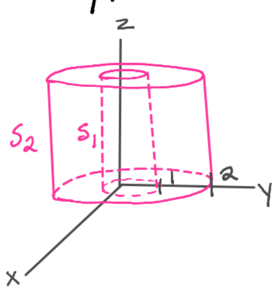
$\rightarrow$ Messy!

Divergence Theorem: $\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle 3y, \frac{2}{7}x^2, \frac{1}{2}z^2 \right\rangle = 2$$

$$\begin{aligned}
\iiint_D \operatorname{div} \vec{F}\, dV &= \iiint_D 2\, dV \\
\text{Spherical} \quad &= \int_0^{2\pi} \int_0^{\pi} \int_1^{2} 2\, (\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta \\
&= 4\pi \left[\int_1^2 \rho^2 d\rho\right] \left[\int_0^{\pi} \sin\varphi\, d\varphi\right] \\
&= 4\pi \left[\tfrac{1}{3}\rho^3\right]_1^2 \left[-\cos\varphi\right]_0^{\pi} \\
&= 4\pi \left[\tfrac{8}{3} - \tfrac{1}{3}\right] [2] \\
&= \boxed{\frac{56\pi}{3}} \quad \blacksquare
\end{aligned}$$

Ex.2 Compute the net outward flux of $\vec{F} = \langle x, 2y, 3z \rangle$ across the boundary of the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ for $0 \le z \le 8$.



S_1: $\vec{r}_1(u,v) = \langle \cos u, \sin u, v \rangle$ $\qquad 0 \le u \le 2\pi$, $0 \le v \le 8$

$\vec{r}_{1,u} = \langle -\sin u, \cos u, v \rangle$

$\vec{r}_{1,v} = \langle 0, 0, 1 \rangle$

$\vec{n} = \langle \cos u, \sin u, 0 \rangle$

S_2: $\vec{r}_2(u,v) = \langle 2\cos u, 2\sin u, v \rangle$ $\qquad 0 \le u \le 2$, $0 \le v \le 8$

$\vec{n}_2 = \langle 2\cos u, 2\sin u, 0 \rangle$

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n}\, dS &= \iint_{S_2} \vec{F} \cdot \vec{n}_2\, dS - \iint_{S_1} \vec{F} \cdot \vec{n}_1\, dS \\
&= \iint_{R_2} (4\cos^2 u + 8\sin^2 u)\, dA \\
&\quad - \iint_{R_1} (\cos^2 u + 2\sin^2 u)\, dA \\
&= \text{Annoying!}
\end{aligned}$$

Divergence Thm: $\nabla \cdot \vec{F} = 1 + 2 + 3 = 6$

$$\iiint_D \nabla \cdot \vec{F} \, dV = \int_0^8 \int_0^{2\pi} \int_1^2 6 \underbrace{r \, dr \, d\theta \, dz}_{\text{cylindrical } dV}$$

$$= [8][2\pi] \left[3r^2\right]_1^2$$

$$= 48\pi \, (4 - 1)$$

$$= \boxed{144\pi} \quad \blacksquare$$