WEIGHT CYCLING AND SUPERSINGULAR REPRESENTATIONS

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Abstract. Let F/F^+ be a CM extension unramified at all finite places such that p is unramified in F^+ and all places $v|p$ of F^+ split in F. Let $\overline{r} : G_F \to \mathrm{GL}_3(\overline{\mathbf{F}}_p)$ be a modular Galois representation satisfying the Taylor–Wiles hypotheses, and is irreducible and generic locally at a place $w|v|p$. Let $\mathfrak{m}$ be the corresponding Hecke eigensystem. Under mild hypotheses, we show that the $\mathfrak{m}$-torsion in the space of mod p modular forms on a compact unitary group over F^+ split over F is an indecomposable $\mathrm{GL}_3(F_w)$-representation. The main ingredients are multiplicity one results of [LLHLM16] and a weight cycling method due to Buzzard and [EGH13].

1. Introduction

Let L be a finite extension of $\mathbf{Q}_p$, and let $\overline{\rho} : G_L \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ be a continuous Galois representation. When $L = \mathbf{Q}_p$ and $n = 2$, Breuil ([Bre03]) proposed a mod p Langlands correspondence which would attach to such a Galois representation a characteristic p $\mathrm{GL}_2(\mathbf{Q}_p)$-representation $\Pi(\overline{\rho})$. This correspondence is combinatorially simple—in particular, if $\overline{\rho}$ is irreducible then so is $\Pi(\overline{\rho})$, and if $\overline{\rho}$ is a (split) extension then so is $\Pi(\overline{\rho})$. This suggested the existence of a functor Π from such Galois representations to $\mathrm{GL}_2(\mathbf{Q}_p)$-representations realizing this correspondence. Such a functor was constructed by Colmez ([Col10]). Shortly thereafter, Emerton ([Eme11]) used this functor to prove local-global compatibility, or in other words, that this correspondence is compatible with the completed cohomology of modular curves. In recent years, a number of candidate functors which also apply when $L \neq \mathbf{Q}_p$ or $n > 2$ have been constructed (see e.g. [Bre15, GK16, Zab]). At present, little is known about these functors, although it is natural to hope that they are compatible with completed cohomology. Breuil, Herzig, and Schraen have suggested that the expected exactness and local-global compatibility properties of these functors should relate the submodule structure of $\overline{\rho}$ to the submodule structure of completed cohomology generalizing the case of $\mathrm{GL}_2(\mathbf{Q})$.

To fix ideas, let F/F^+ be a CM extension and let $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ be an automorphic Galois representation such that there are places $w|v|p$ of F and F^+ such that v splits over F, F_w is isomorphic to L, and $\overline{r}|_{G_{F_w}}$ is isomorphic to $\overline{\rho}$. Then we can define a $\mathrm{GL}_n(L)$-representation Π_{glob} from the space of mod p automorphic forms on a definite unitary group over F^+ which splits over F, which serves as a candidate for the mod p Langlands correspondence applied to $\overline{\rho}$ (see §4). We can moreover choose level and coefficients so that Π_{glob} satisfies natural minimality properties. The following natural conjecture relates the submodule structures of $\overline{\rho}$ and Π_{glob} in a special case.

Conjecture 1.1. *Suppose that L is unramified and $\overline{\rho}$ is irreducible and generic. Then Π_{glob} is an irreducible $\mathrm{GL}_n(L)$-representation.*

In the $n = 2$ case, [BP12] and [EGS15] show that the $\mathrm{GL}_2(L)$-socle of Π_{glob} is irreducible. The following is our main result.

Theorem 1.2. *Suppose that L is an unramified extension of $\mathbf{Q}_p$, $n = 3$, and $\overline{\rho}$ is irreducible and 10-generic. Suppose further that F/F^+ is unramified at all finite places, $\overline{r}$ satisfies the Taylor–Wiles hypotheses, and $\overline{r}$ is Fontaine–Laffaille at all places dividing p. Then Π_{glob} is an indecomposable $\mathrm{GL}_n(L)$-representation.*

With the additional hypothesis that L is $\mathbf{Q}_p$, it can even be shown that the $\mathrm{GL}_3(\mathbf{Q}_p)$-socle of Π_{glob} is irreducible from the mod p multiplicity one results of [LLHLM16] and the techniques of [EGS15]. We briefly sketch the argument here. Suppose that π' is a nonzero irreducible subrepresentation of Π_{glob} with $\mathrm{GL}_3(\mathcal{O}_L)$-socle containing the Serre weight (irreducible $\mathrm{GL}_3(\mathcal{O}_L)$-representation) σ. Let I_1 and I be the subgroups of $\mathrm{GL}_3(\mathcal{O}_L)$ whose elements are upper triangular unipotent and upper triangular modulo p, respectively. Then the space of I_1-invariants of σ is one-dimensional. Suppose that I acts on this space by the character χ. The matrices

$$\begin{pmatrix} & 1 & \\ & & 1 \\ p & & \end{pmatrix} \text{ and } \begin{pmatrix} & & 1 \\ p & & \\ & p & \end{pmatrix}$$

normalize I_1 so that they cyclically permute the spaces where I acts by a cyclic permutation of χ. By Frobenius reciprocity, one obtains nonzero maps from principal series types $c_w : \mathrm{Ind}_I^{\mathrm{GL}_3(\mathcal{O}_L)} w\chi$ to π' where w is a cyclic permutation. In fact, it can be shown that these maps factor through parabolic inductions which are quotients of these types. In every case, one can use p-adic Hodge theory and characteristic zero automorphic forms to show that for at least one normalizer of I_1 at most two Jordan–Hölder factors σ and $\delta(\sigma)$ of these parabolic inductions is in the $\mathrm{GL}_3(\mathcal{O}_L)$-socle of Π_{glob}. The image of c_w cannot contain σ as a Jordan–Hölder factor because Π_{glob} is known to be a supersingular representation by classical (characteristic 0) local-global compatibility. Thus the $\mathrm{GL}_3(\mathcal{O}_L)$-socle of the image of a nonzero map $\mathrm{Ind}_I^{\mathrm{GL}_3(\mathcal{O}_L)} w\chi \to \pi'$ contains $\delta(\sigma)$. This procedure is known as weight cycling. Repeating this weight cycling procedure and using mod p multiplicity one, one shows that the $\mathrm{GL}_3(\mathcal{O}_L)$-socles of π' and Π_{glob} coincide. Thus π' must be the G-socle of Π_{glob}.

There are two major issues with generalizing this result to cases where L is not $\mathbf{Q}_p$. The first issue is that usually more than two of the Jordan–Hölder factors of the relevant parabolic inductions lie in the $\mathrm{GL}_3(\mathcal{O}_L)$-socle of Π_{glob}. Fortunately, results in [LLHLM16] show that the dimension of the space of maps $\mathrm{Ind}_I^{\mathrm{GL}_3(\mathcal{O}_L)} w\chi \to \Pi_{\mathrm{glob}}$ is often one-dimensional, from which the image of such a map can be determined. This fully describes the weight cycling procedure in many cases. The second issue is that the weight cycling procedure is not transitive on the $\mathrm{GL}_3(\mathcal{O}_L)$-socle of Π_{glob} when L is not $\mathbf{Q}_p$. There is a partial solution to this problem which was described in [BP12] when $n = 2$. Instead of starting with an I_1-invariant of the $\mathrm{GL}_3(\mathcal{O}_L)$-socle of Π_{glob}, one can start with *any* I_1-invariant and apply a normalizer of I_1. If we assume that π' is a *direct summand* of Π_{glob} containing a Serre weight σ in its $\mathrm{GL}_3(\mathcal{O}_L)$-socle, then π' contains any I_1-invariant whose $\mathrm{GL}_3(\mathcal{O}_L)$-span contains σ. This produces further I_1-invariants of π' to which we can apply the weight cycling procedure.

It is not clear to the author whether this refined weight cycling procedure is transitive on the $\mathrm{GL}_3(\mathcal{O}_L)$-socle of Π_{glob} because of the rather different behavior exhibited by so-called obvious and shadow weights. This contrast between obvious and shadow weights is also exhibited in the description (refining the mod p multiplicity results of [LLHLM16]) of the invariants of Π_{glob} under the first principal congruence subgroup in [LLHM17]. Fortunately, this description is exactly what is needed to finish the proof that the $\mathrm{GL}_3(\mathcal{O}_L)$-socles of π' and Π_{glob} coincide. This implies that π' and Π_{glob} coincide.

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1.2. **Notation.** If F is any field, we write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$, where $\overline{F}$ is a separable closure of F. Let $L/\mathbf{Q}_p$ be an unramified extension of degree f. Let $q = p^f$ and identify $\mathbf{F}_q$ with the residue field of L. Let I_L be the inertial subgroup of G_L.

Let E be a finite extension of L with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$. Fix an embedding $\sigma_0 : \mathbf{F}_q \hookrightarrow \mathbf{F}$. Let φ denote the p-th power Frobenius on $\mathbf{F}_q$. For $1 \le i \le f-1$, let $\sigma_i : \mathbf{F}_q \hookrightarrow \mathbf{F}$ be $\sigma_0 \circ \varphi^{-i}$.

Let $\underline{G}_0$ be $\mathrm{Res}_{\mathbf{F}_{p^f}/\mathbf{F}_p} \mathrm{GL}_3$ and let $\underline{G}$ be $\underline{G}_0 \times_{\mathbf{F}_p} \mathbf{F}$. Let $\underline{W}$ be the Weyl group of $\underline{G}$. Let $\underline{T}$ be the diagonal torus of $\underline{G}$ and $X^*(\underline{T})$ its group of characters. Let $\underline{\widetilde{W}}$ be the extended affine Weyl group for $\underline{G}$. There is an isomorphism

$$\underline{G} \cong \prod_{i \in \mathbf{Z}/f} \mathrm{GL}_{3/\mathbf{F}}$$

where the i-th coordinate corresponds to σ_{-i}. There is a similar decomposition for $\underline{W}$, $\underline{T}$, $X^*(\underline{T})$, and $\underline{\widetilde{W}}$. Let C_0 be the set of dominant characters $\omega \in X^*(\underline{T})$ such that $\langle \omega, \alpha \rangle < p-2$ for all simple positive roots α.

Let $\eta' \in X^*(\underline{T})$ be the lift of the half sum of the positive roots which is $(2,1,0)$ in every factor.

2. Patching functors and Serre weights

Let $L/\mathbf{Q}_p$ be an unramified extension and $\overline{\rho} : G_L \to \mathrm{GL}_3(\mathbf{F})$ be an irreducible 10-generic Galois representation. Then there is a $\mu \in C_0$ such that $\overline{\rho}|_{I_L} = \tau(s, \mu+\underline{1})$ where $s = (1,1,\dots,1,w) \in \underline{W}$, $w = (123)$ or (132), and we use the parametrization of inertial types for GL_n in [GHS15, §9.2]. Recall the definition of $\mathfrak{Tr}_\mu$ from [LLHLM16, §2.1]. We recall the definitions of Σ_0, Σ_0^{obv} and Σ from [LLHLM16, §2.2]. We have the following result which follows from [LLHLM16, Prop. 2.2.6].

Proposition 2.1. *Then* $W^?(\overline{\rho}) = F(\mathfrak{Tr}_\mu(sr(\Sigma)))$.

We recall the notion of a weak minimal patching functor for $\overline{\rho}$. Let $R^\square_{\overline{\rho}}$ be the (unrestricted) universal $\mathcal{O}$-framed deformation ring of $\overline{\rho}$. Fix a nonnegative integer h and let R_∞ be $R^\square_{\overline{\rho}}[\![x_1,\dots,x_h]\!]$. For an inertial type τ of G_L, let $R^\tau_{\overline{\rho}}$ be the universal $\mathcal{O}$-framed deformation ring of $\overline{\rho}$ parameterizing lifts of p-adic Hodge type $((2,1,0),\tau)$. Let $R_\infty(\tau)$ be $R^\tau_{\overline{\rho}} \otimes_{R^\square_{\overline{\rho}}} R_\infty$. Let $K \subset G$ be $\mathrm{GL}_3(\mathcal{O}_L) \subset \mathrm{GL}_3(L)$, respectively. For an inertial type τ of G_L, we also denote by *tau* the corresponding

K-representation under the inertial local Langlands correspondence (see [CEG$^+$14, Theorem 3.7]).

Definition 1. A *weak minimal patching functor for $\overline{\rho}$* is defined to be a nonzero covariant exact functor $M_\infty : \mathrm{Rep}_K(\mathcal{O}) \to R_\infty$-Mod satisfying the following axioms:

(1) Let τ be an inertial type for G_L. If τ° is an $\mathcal{O}$-lattice in the K-representation τ, then $M_\infty(\sigma(\tau)^\circ)$ is p-torsion free and is maximally Cohen–Macaulay over $R_\infty(\tau)$;
(2) if σ is an irreducible $\mathrm{GL}_3(\mathbf{F}_q)$-representations over $\mathbf{F}$, the module $M_\infty(\sigma)$ is either 0 or Cohen–Macaulay of length d; and
(3) the locally free sheaf $M_\infty(\tau^\circ)[1/p]$ (being maximal Cohen–Macaulay over the regular generic fiber of $X_\infty(\tau)$) has rank at most one on each connected component.

Let M_∞ be a weak minimal patching functor for $\overline{\rho}$ that has an action of G commuting with the action of $R_\infty[\![K]\!]$. Let π be the $\mathrm{GL}_3(L)$-representation $(M_\infty/\mathfrak{m})^\vee$.

Theorem 2.2.
$$\mathrm{soc}_K \pi \cong \oplus_{\sigma \in W^?(\overline{\rho})} \sigma.$$

Proof. [LLHLM16, Theorem 3.4.3 and Lemma 5.2.2]. □

3. Weight cycling

Recall that to a tame inertial type, one can associate an element $\widetilde{w}^*(\overline{\rho},\tau) \in \underline{\widetilde{W}}$ as in [LLHL16, §4]. In this section, we let τ be a generic tame inertial type such that $\widetilde{w}^*(\overline{\rho},\tau)$ is η'-admissible. Then $\overline{\rho}$ has a potentially crystalline lift of type τ and parallel Hodge–Tate weights $(2,1,0)$. Then there is a well-defined element $\widetilde{w}(\overline{\rho},\tau) \in \underline{\widetilde{W}}$ as in [LLHLM17, Definition 3.3]. Assume that $\ell(\widetilde{w}(\overline{\rho},\tau)_i)$ is even for all i. Recall the definitions of M_∞ and π from §2.

In this section, we fix a weight $\sigma = F(\mathfrak{Tr}(\omega,a)) \in \mathrm{JH}(\overline{\tau})$ such that $(\omega_i,a_i) \in s_i\widetilde{w}^*(\overline{\rho},\tau)_i^{-1}(\Sigma_0^{\mathrm{obv}})$ for all i. If $\ell(\widetilde{w}(\overline{\rho},\tau)_i) = 2$, there is a unique $(\omega_i^0,a_i^0) \in s_i\widetilde{w}(\overline{\rho},\tau)_i^{-1}(\Sigma_0) \cap s_i r(\Sigma_0)$ such that

$$\mathrm{d}_{\mathrm{gph}}((\omega_i,a_i),(\omega_i^0,a_i^0)) \le \mathrm{d}_{\mathrm{gph}}((\omega_i,a_i),(\omega_i',a_i')) \tag{3.1}$$

for all $(\omega_i',a_i') \in s_i\widetilde{w}(\overline{\rho},\tau)_i^{-1}(\Sigma_0) \cap s_i r(\Sigma_0)$ with equality if and only if $(\omega_i',a_i') = (\omega_i^0,a_i^0)$.

Proposition 3.1. *Suppose that $\sigma' = F(\mathfrak{Tr}(\omega',a')) \in W^?(\overline{\rho},\tau)$ and that for some $j \in \mathbf{Z}/f$, either*

(1) $\ell(\widetilde{w}(\overline{\rho},\tau)_j) = 2$ *and* $(\omega_j',a_j') \ne (\omega_j^0,a_j^0)$ *or*
(2) $\ell(\widetilde{w}(\overline{\rho},\tau)_j) = 0$ *and* $(\omega_j',a_j') \in s_j r(\Sigma_0^{\mathrm{obv}})$.

Then σ' is not a Jordan–Hölder factor of the image of any map $\overline{\tau}^\sigma \to \pi$.

Proof. Let $\sigma'' = F(\mathfrak{Tr}(\omega'',a'')) \in W^?(\overline{\rho},\tau)$ such that

- $(\omega_i'',a_i'') = (\omega_i^0,a_i^0)$ if $\ell(\widetilde{w}(\overline{\rho},\tau)_i) = 2$
- $(\omega_i'',a_i'') \notin s_i r(\Sigma_0^{\mathrm{obv}})$ if $\ell(\widetilde{w}(\overline{\rho},\tau)_i) = 0$ and
$$\mathrm{d}_{\mathrm{gph}}((\omega_i'',a_i''),(\omega_i,a_i)) \le \mathrm{d}_{\mathrm{gph}}((\omega_i',a_i'),(\omega_i,a_i))$$
with equality if and only if $(\omega_i'',a_i'') = (\omega_i',a_i')$.

Then any map $\overline{\tau}^{\sigma'} \to \overline{\tau}^{\sigma}$ factors as $\overline{\tau}^{\sigma'} \to \overline{\tau}^{\sigma''} \to \overline{\tau}^{\sigma}$ by [LLHLM16, Theorem 4.1.2], and $M_\infty(\overline{\tau}^{\sigma''})$ is a cyclic R_∞-module by [LLHLM16, Theorem 5.2.1]. With the hypotheses of the proposition, any map $M_\infty(\overline{\tau}^{\sigma'}) \to M_\infty(\overline{\tau}^{\sigma''})$ has nontrivial cokernel. Thus any map $\overline{\tau}^{\sigma'} \to \overline{\tau}^{\sigma}$ induces a map $M_\infty(\overline{\tau}^{\sigma'}) \to M_\infty(\overline{\tau}^{\sigma})$ which factors through $\mathfrak{m}M_\infty(\overline{\tau}^{\sigma''})$ and thus $\mathfrak{m}M_\infty(\overline{\tau}^{\sigma})$. ☐

Let K_1 be the kernel of the natural reduction map $K \to \mathrm{GL}_3(\mathbf{F}_q)$).

Proposition 3.2. *Suppose that $\pi' \subset \pi^{K_1}$ is a K-direct summand. Let $\sigma^1 = F(\mathfrak{Tr}((\omega^1, a^1)))$ and $\sigma^2 = F(\mathfrak{Tr}((\omega^2, a^2))) \in W^?(\overline{\rho})$ such that $(\omega_i^1, a_i^1) \in s_j r(\Sigma_0^{\mathrm{obv}})$ if and only if $(\omega_i^2, a_i^2) \in s_j r(\Sigma_0^{\mathrm{obv}})$ and $(\omega_i^1, a_i^1) = (\omega_i^2, a_i^2)$ if $(\omega_i^1, a_i^1) \in s_j r(\Sigma_0^{\mathrm{obv}})$. Then $\mathrm{Hom}_K(\sigma^1, \pi')$ is nonzero if and only if $\mathrm{Hom}_K(\sigma^2, \pi')$ is nonzero.*

Proof. This follows from [LLHM17, Theorem 4.19]. ☐

Let $W^?(\overline{\rho}, \tau)$ be the intersection $W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\tau})$.

Proposition 3.3. *Suppose that $\pi' \subset \pi^{K_1}$ is a K-direct summand. Let $\sigma' = F(\mathfrak{Tr}(\omega', a'))$ be a weight in $W^?(\overline{\rho}, \tau)$ such that*

- *if $\ell(\widetilde{w}(\overline{\rho}, \tau)_i) = 2$ then $(\omega_i', a_i') = (\omega_i^0, a_i^0)$; and*
- *if $\ell(\widetilde{w}(\overline{\rho}, \tau)_i) = 0$ then $(\omega_i', a_i') \notin s_i r(\Sigma_0^{\mathrm{obv}})$.*

Then $\mathrm{Hom}_K(\overline{\tau}^{\sigma}, \pi')$ is nonzero if and only if $\mathrm{Hom}_K(\sigma', \pi')$ is nonzero.

Proof. Suppose that $\mathrm{Hom}_K(\overline{\tau}^{\sigma}, \pi')$ is nonzero. By Proposition 3.1, if $\overline{\tau}^{\sigma} \to \pi'$ is a nonzero map, then the K-socle of its image must contain a weight satisfying the itemized properties in the proposition. By Proposition 3.2, the K-socle of π' contains one weight satisfying the itemized properties in the proposition if and only if it contains all such weights. Thus, $\mathrm{Hom}_K(\sigma', \pi')$ is nonzero.

Suppose that $\mathrm{Hom}_K(\sigma', \pi')$ is nonzero. By Proposition 3.2, the $\mathrm{soc}_K\pi/\mathrm{soc}_K\pi'$ contains no weights satisfying the itemized properties in the proposition. Note that since $M_\infty(\sigma')$ is nonzero, so is $M_\infty(\overline{\tau}^{\sigma})$ and $\mathrm{Hom}_K(\overline{\tau}^{\sigma}, \pi)$. The K-socle of the image of a nonzero map $\theta : \overline{\tau}^{\sigma} \to \pi$ contains only weights satisfying the itemized properties in the proposition, and is therefore in $\mathrm{soc}_K\pi'$. Since $\pi' \subset \pi^{K_1}$ is a K-direct summand, π' contains the image of θ. Thus $\mathrm{Hom}_K(\overline{\tau}^{\sigma}, \pi')$ is nonzero. ☐

For a generic tame inertial type τ, let $w(\overline{\rho}, \tau)$ be the image of $\widetilde{w}(\overline{\rho}, \tau)$ in $\underline{W}$. Recall the definition of w from §2.

Lemma 3.4. *Suppose that $\pi' \subset \pi$ is a G-subrepresentation such that $\pi'^{K_1} \subset \pi^{K_1}$ is a K-direct summand. Let τ be a generic tame type such that for some $j \in \mathbf{Z}/f$, $w(\overline{\rho}, \tau)_i = 1$ for $i \neq j$ and $w(\overline{\rho}, \tau)_j = w^{-1}$. Then for σ and $\sigma' \in W^?(\overline{\rho}, \tau)$, $\mathrm{Hom}_K(\sigma, \pi')$ is nonzero if and only if $\mathrm{Hom}_K(\sigma', \pi')$ is nonzero.*

Proof. From [Her09, Lemma 4.2], we see that τ is a principal series type. We note that if $i \neq j$ and $(\omega_i, a_i) \in \Sigma_0^{\mathrm{obv}}$ (resp. $(\omega_i, a_i) \notin \Sigma_0^{\mathrm{obv}}$), then $\ell(\widetilde{w}(\overline{\rho}, \tau)_i) = 4$ (resp. $\ell(\widetilde{w}(\overline{\rho}, \tau)_i) = 0$, while $\ell(\widetilde{w}(\overline{\rho}, \tau)_j) = 2$. Let $I \subset K$ (resp. $I_1 \subset K$) be the standard Iwahori subgroup (resp. pro-p Iwahori subgroup) which is the preimage of upper triangular matrices (resp. upper triangular unipotent matrices) under the reduction map. Then $\tau \cong \mathrm{Ind}_I^K \chi$ for six possible characters χ. Let $\sigma_\chi = F(\mathfrak{Tr}(\omega, a))$ be the unique Serre weight such that $\sigma^{I_1} \cong \chi$. The six possible choices of χ give the six possible choices for $(\omega_j, a_j) \in s_j\widetilde{w}(\overline{\rho}, \tau)_j^{-1}(\Sigma_0^{\mathrm{obv}})$. Fix one of the two choices of ω_j^0 such that $(\omega_j^0, 0) \in s_i\widetilde{w}(\overline{\rho}, \tau)_i^{-1}(\Sigma_0) \cap s_i r(\Sigma_0)$, and let $(\omega_j, a_j) \in s_j\widetilde{w}(\overline{\rho}, \tau)_j^{-1}(\Sigma_0^{\mathrm{obv}})$

be the unique element such that (3.1) is satisfied. Then by Proposition 3.3, ${\pi'}^{I_1,\chi}$ is nonzero if and only if $\mathrm{Hom}_K(\sigma, \pi')$ is nonzero for any $\sigma = F(\mathfrak{Tr}(\omega, a)) \in W^?(\overline{\rho}, \tau)$ with $(\omega_j, a_j) = (\omega_j^0, 0)$.

Using normalizers of the Iwahori, we see that the nonvanishing of ${\pi'}^{I_1,\chi}$, ${\pi'}^{I_1,(123)\chi}$, and ${\pi'}^{I_1,(132)\chi}$ are equivalent. Again by Proposition 3.3, ${\pi'}^{I_1,(123)\chi}$ and ${\pi'}^{I_1,(132)\chi}$ are nonzero if and only if $\mathrm{Hom}_K(\sigma', \pi')$ is nonzero for any $\sigma' = F(\mathfrak{Tr}(\omega', a')) \in W^?(\overline{\rho}, \tau)$ with $a'_j = 1$. Combining this with Proposition 3.2, the nonvanishing of $\mathrm{Hom}_K(\sigma, \pi')$ is equivalent for all $\sigma \in W^?(\overline{\rho}, \tau)$. □

Theorem 3.5. *Suppose that $\pi' \subset \pi$ is a nonzero G-subrepresentation such that ${\pi'}^{K_1} \subset \pi^{K_1}$ is a K-direct summand. Then π' is indecomposable as a G-representation. In particular, π is indecomposable as a G-representation.*

Proof. Let $\pi'' \subset \pi'$ be a nonzero G-direct summand. It suffices to show that the nonvanishing of $\mathrm{Hom}_K(\sigma, \pi'')$ and $\mathrm{Hom}_K(\sigma', \pi'')$ are equivalent for $\sigma = F(\mathfrak{Tr}(sr(\omega, a)))$ and $\sigma' = F(\mathfrak{Tr}(sr(\omega', a')))$ in $W^?(\overline{\rho})$. Inducting on $k \stackrel{\text{def}}{=} \#\{i : (\omega_i, a_i) \neq (\omega'_i, a'_i)\}$ and possibly relabeling, we can assume without loss of generality that $k = 1$. Suppose that $(\omega_j, a_j) \neq (\omega'_j, a'_j)$. There is a generic tame inertial type τ (resp. τ') such that $\sigma \in \mathrm{JH}(\overline{\tau})$ (resp. $\sigma' \in \mathrm{JH}(\overline{\tau}')$), $w(\overline{\rho}, \tau)_i = 1$ (resp. $w(\overline{\rho}, \tau')_i = 1$) for $i \neq j$ and $w(\overline{\rho}, \tau)_j = w^{-1}$ (resp. $w(\overline{\rho}, \tau')_j = w^{-1}$). One can check that $W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\tau}) \cap \mathrm{JH}(\overline{\tau}')$ is nonempty, and so the nonvanishing of $\mathrm{Hom}_K(\sigma, \pi'')$ and $\mathrm{Hom}_K(\sigma', \pi'')$ are equivalent by Lemma 3.4 applied to $\pi'' \subset \pi$. □

4. Global applications

We now apply §3 to a global context. We follow [LLHLM17, §7.1] and refer the reader there for any unfamiliar notation or terminology. Let F be a CM field with maximal totally real field $F^+ \neq \mathbf{Q}$ in which p is unramified. Suppose that the extension F/F^+ is unramified at all finite places and that all places of F^+ dividing p split in F. Let $G_{/F^+}$ be an outer form for GL_3 which is compact at infinity and quasisplit at all finite places of F^+ and which splits over F. Then as in [LLHLM17, §7.1], we define spaces $S(U, W)$ of algebraic automorphic forms of level U and coefficients W. For a place $v|p$, let $S(U^v, W)$ be

$$\varinjlim_{U_v} S(U^v U_v, W)$$

where U_v ranges over compact open subgroups of $G(F_v^+)$. Suppose that $\overline{r} : G_F \to \mathrm{GL}_3(\mathbf{F})$ is a modular representation which is Fontaine–Laffaille at all places dividing p and there is a place $w|p$ of F such that $\overline{\rho} \stackrel{\text{def}}{=} \overline{r}|_{G_{F_w}}$ is an irreducible 8-generic Galois representation. Let v be the place $w|_{F^+}$. Let W be as in [LLHLM16, §5.4.1]. Let $\mathfrak{m}$ be the kernel of the system of Hecke eigenvalues associated to $\overline{r}$ as in [LLHLM17, §7.1]. For each place $v'|p$ not equal to v, there is an element $\underline{a}_{v'} = \{\underline{a}_{w'}, \underline{a}_{w'^c}\} \in (X_1^{(3)})_v$ such that $0 \leq \langle \underline{a}_{w'}, \alpha \rangle$ for all positive simple roots α and $S(U, \otimes_{v' \neq v} F_{\underline{a}_{v'}} \otimes W)_{\mathfrak{m}}$ is nonzero.

Let U^v be the maximal compact subgroup of $G(\mathbf{A}_{F^+}^{v,\infty})$ which is hyperspecial for every factor. Let Π_{glob} be $S(U^v, \otimes_{v' \neq v} F_{\underline{a}_{v'}} \otimes W)[\mathfrak{m}]$ which has a natural action of $\mathrm{GL}_3(F_w)$ through ι_w.

Proof of Theorem 1.2. As in [LLHLM16, Prop. 3.4.14], one can construct a weak minimal patching functor M_∞ for $\overline{\rho}$ such that $M_\infty^\vee[\mathfrak{m}]$ is naturally isomorphic to Π_{glob}. Then Theorem 3.5 applies with π' and π both taken to be Π_{glob}. □

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