Section 2.2 Absolute Value and the Real Line

From the Trichotomy Property 2.1.5(iii), we are assured that if $a \in \mathbb{R}$ and $a \neq 0$, then exactly one of the numbers a and $-a$ is positive. The absolute value of $a \neq 0$ is defined to be the positive one of these two numbers. The absolute value of 0 is defined to be 0.

2.2.1 Definition The **absolute value** of a real number a, denoted by $|a|$, is defined by

$$|a| := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

For example, $|5| = 5$ and $|-8| = 8$. We see from the definition that $|a| \geq 0$ for all $a \in \mathbb{R}$, and that $|a| = 0$ if and only if $a = 0$. Also $|-a| = |a|$ for all $a \in \mathbb{R}$. Some additional properties are as follows.

2.2.2 Theorem **(a)** $|ab| = |a||b|$ *for all* $a, b \in \mathbb{R}$.

(b) $|a|^2 = a^2$ *for all* $a \in \mathbb{R}$.

(c) *If* $c \geq 0$, *then* $|a| \leq c$ *if and only if* $-c \leq a \leq c$.

(d) $-|a| \leq a \leq |a|$ *for all* $a \in \mathbb{R}$.

Proof. (a) If either a or b is 0, then both sides are equal to 0. There are four other cases to consider. If $a > 0, b > 0$, then $ab > 0$, so that $|ab| = ab = |a||b|$. If $a > 0, b < 0$, then $ab < 0$, so that $|ab| = -ab = a(-b) = |a||b|$. The remaining cases are treated similarly.

(b) Since $a^2 \geq 0$, we have $a^2 = |a^2| = |aa| = |a||a| = |a|^2$.

(c) If $|a| \leq c$, then we have both $a \leq c$ and $-a \leq c$ (why?), which is equivalent to $-c \leq a \leq c$. Conversely, if $-c \leq a \leq c$, then we have both $a \leq c$ and $-a \leq c$ (why?), so that $|a| \leq c$.

(d) Take $c = |a|$ in part (c). Q.E.D.

The following important inequality will be used frequently.

2.2.3 Triangle Inequality *If* $a, b \in \mathbb{R}$, *then* $|a + b| \leq |a| + |b|$.

Proof. From 2.2.2(d), we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. On adding these inequalities, we obtain

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Hence, by 2.2.2(c) we have $|a + b| \leq |a| + |b|$. Q.E.D.

It can be shown that equality occurs in the Triangle Inequality if and only if $ab > 0$, which is equivalent to saying that a and b have the same sign. (See Exercise 2.)

There are many useful variations of the Triangle Inequality. Here are two.

2.2.4 Corollary *If* $a, b \in \mathbb{R}$, *then*

(a) $\big||a| - |b|\big| \leq |a - b|$,

(b) $|a - b| \leq |a| + |b|$.

Proof. (a) We write $a = a - b + b$ and then apply the Triangle Inequality to get $|a| = |(a - b) + b| \leq |a - b| + |b|$. Now subtract $|b|$ to get $|a| - |b| \leq |a - b|$. Similarly, from $|b| = |b - a + a| \le |b - a| + |a|$, we obtain $-|a - b| = -|b - a| \le |a| - |b|$. If we combine these two inequalities, using 2.2.2(c), we get the inequality in (a).

(b) Replace b in the Triangle Inequality by $-b$ to get $|a - b| \le |a| + |-b|$. Since $|-b| = |b|$ we obtain the inequality in (b). Q.E.D.

A straightforward application of Mathematical Induction extends the Triangle Inequality to any finite number of elements of $\mathbb{R}$.

2.2.5 Corollary *If $a_1, a_2, \ldots, a_n$ are any real numbers, then*

$$|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|.$$

The following examples illustrate how the properties of absolute value can be used.

2.2.6 Examples **(a)** Determine the set A of $x \in \mathbb{R}$ such that $|2x + 3| < 7$.

From a modification of 2.2.2(c) for the case of strict inequality, we see that $x \in A$ if and only if $-7 < 2x + 3 < 7$, which is satisfied if and only if $-10 < 2x < 4$. Dividing by 2, we conclude that $A = \{x \in \mathbb{R} : -5 < x < 2\}$.

(b) Determine the set $B := \{x \in \mathbb{R} : |x - 1| < |x|\}$.

One method is to consider cases so that the absolute value symbols can be removed. Here we take the cases

$$\text{(i) } x \ge 1, \quad \text{(ii) } 0 \le x < 1, \quad \text{(iii) } x < 0.$$

(Why did we choose these three cases?) In case (i) the inequality becomes $x - 1 < x$, which is satisfied without further restriction. Therefore all x such that $x \ge 1$ belong to the set B. In case (ii), the inequality becomes $-(x - 1) < x$, which requires that $x > \frac{1}{2}$. Thus, this case contributes all x such that $\frac{1}{2} < x < 1$ to the set B. In case (iii), the inequality becomes $-(x - 1) < -x$, which is equivalent to $1 < 0$. Since this statement is false, no value of x from case (iii) satisfies the inequality. Forming the union of the three cases, we conclude that $B = \left\{x \in \mathbb{R} : x > \frac{1}{2}\right\}$.

There is a second method of determining the set B based on the fact that $a < b$ if and only if $a^2 < b^2$ when both $a \ge 0$ and $b \ge 0$. (See 2.1.13(a).) Thus, the inequality $|x - 1| < |x|$ is equivalent to the inequality $|x - 1|^2 < |x|^2$. Since $|a|^2 = a^2$ for any a by 2.2.2(b), we can expand the square to obtain $x^2 - 2x + 1 < x^2$, which simplifies to $x > \frac{1}{2}$. Thus, we again find that $B = \left\{x \in \mathbb{R} : x > \frac{1}{2}\right\}$. This method of squaring can sometimes be used to advantage, but often a case analysis cannot be avoided when dealing with absolute values.

A graphical view of the inequality is obtained by sketching the graphs of $y = |x|$ and $y = |x - 1|$, and interpreting the inequality $|x - 1| < |x|$ to mean that the graph of $y = |x - 1|$ lies underneath the graph of $y = |x|$. See Figure 2.2.1.

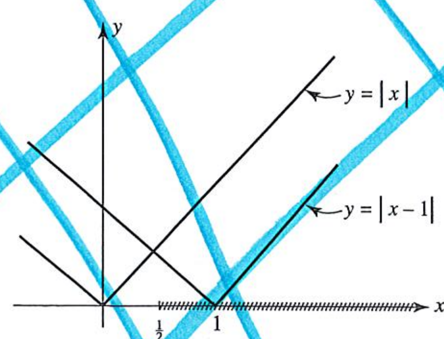

Figure 2.2.1 $|x - 1| < |x|$