HW 4-18-23

① Consider $\Omega = \{1, 2, \dots, m\}$, $\mathcal{A} = \{\text{all subsets of } \Omega\}$ and μ is counting measure. Show that $(L^2(\Omega, \mathcal{A}, \mu), \| \|_2)$ and $(\mathbb{R}^m$, Euclidean norm$)$ are isometrically isomorphic.

② If $a, b \in \mathbb{R}$, $a < b$, let $C^1[a,b]$ consist of differentiable functions $f : [a,b] \to \mathbb{R}$, whose derivative is continuous. Let $\|f\| = \max_{[a,b]} |f| + \max_{[a,b]} |f'|$. Show $(C^1[a,b], \| \|)$ is a Banach space. Check only two of the axioms; one can be any of your choice, the other should be completeness, though.

③ If $1 \le p \le \infty$, construct $\varphi_j \in L^p[0,1]$, $j \in \mathbb{N}$, such that $\|\varphi_j\|_p \le 1$ $\forall j$ and $\|\varphi_j - \varphi_k\|_p \ge 1$ for all $j \ne k$. (From this sequence you cannot select a convergent subsequence; therefore the closed unit ball of $L^p[0,1]$ is not compact.)

HW 4-~~1~~9-23

① If $f \in L^p[a,b]$, $a \le x \le y \le b$, show $\left| \int_x^y f \right| \le C |y-x|^{1/q}$, with some C independent of x, y, and $\frac{1}{p} + \frac{1}{q} = 1$.

② If $(\Omega, \mathcal{A}, \mu)$ is a measure space, how would you define $e^{(\Omega, \mathcal{A}, \mu)}$? (Should also be a measure space.)