1. Motivation and Introduction

It’s well known that if the vector field $b(t,x)$ is Lipschitz in $\mathbb{R}^n$, then the ODE

\[
\frac{dX}{dt} = b(X), \quad X(0,x) = x
\]

has a unique Lipschitz continuous solution continuously depending on the initial data $x$. In these standard results, measure theory plays no role. It has been a permanent question to extend any part of this elementary theory to less regular vector fields $b$. In some situations one might hope for a “generic” uniqueness of the solution of ODE, i.e., for “almost every” initial datum $x$. An even weaker requirement is the research of a "selection principle", i.e., a strategy to select for almost every $x$ a solution $X(\cdot,x)$ in such a way that this selection is stable w.r.t smooth approximations of $b$. In other words, I would like to know that, whenever we approximate $b$ by smooth vector fields $b_h$, the classical trajectories $X^h$ associated to $b^h$ satisfy

\[
\lim_{h \to \infty} X^h(\cdot,x) = X(\cdot,x) \quad \text{in } C([0,T];\mathbb{R}^N), \text{ for a.e. } x.
\]

The existence, uniqueness and stability of solutions of an equation are usually called the well-postedness of the equation.

First of all, to show the existence of solution of the ODE associated with weakly differentiable $b$, a natural starting point is the following standard argument:

Let $\rho_\epsilon$ be the standard mollifier and $b_\epsilon = b * \rho_\epsilon$. By classical results, there exits unique $X_\epsilon(t,x)$ solving the ODE

\[
\frac{dX_\epsilon}{dt} = b_\epsilon(X_\epsilon), \quad X_\epsilon(0,x) = x,
\]

or equivalently

\[
X_\epsilon(t,x) = x + \int_0^t b_\epsilon(X_\epsilon(s,x))ds
\]

If $b$ is in $L^p$, by Minkowski integral inequality, one can see $X_\epsilon$ is bounded in $L^p$ in space variable, and absolutely continuous in $t$ variable, uniformly for $\epsilon$. If we could show (up to a subsequence) $X_\epsilon$ converge to some $X$ in measure, then by using (1.15) and the fact that uniform continuous function is dense in $L^p$ space, one could immediately show for every $t$, up to a subsequence $b_\epsilon(X_\epsilon)$ converges to $b(X)$ in $L^p$, and hence passing the limit in (1.3), we could show for almost every $x$, $X(t,x)$ is absolutely continuous with respect to $t$, solving the ODE (1.1).

So the key question is, what condition we impose on $b$ can guarantee $X_\epsilon$ converges to some $X$ in measure? 

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I thought about this for quite a while, and it seemed to be quite hard to prove this fact by just manipulating the ODE. Later on I learned this subject had already been studied, and the first seminal result is the paper by DiPerna and Lions [1], who deal with the case when \( b \) is Sobolev with bounded divergence in space. They use the well-postedness of transport equation to prove \( X_\epsilon \) converges to some \( X \) in measure, and hence the existence is proven. The stability of the solution of ODE can also be implied by the well-postedness of transport equation, while the uniqueness is proved for \( X \) merely satisfying the ODE in the weak renormalized sense. (See part III of [1]). As far as I know, their idea of using PDE to study ODE motivates the later development of this topic.

Let me briefly show how \( X_\epsilon \) converges in measure can be proved. By standard measure theory, \( X_\epsilon \) converges in measure is equivalent to \( v_\epsilon := \beta(X_\epsilon) \) converges in measure for any bounded \( C^1 \) function \( \beta \), with \( \beta' \) also bounded. It’s not hard to see \( v_\epsilon(t,x) \) satisfies

\[
\partial v_\epsilon \frac{\partial}{\partial t} + b \cdot \nabla v_\epsilon = 0, \tag{1.4}
\]

while \( v_\epsilon(0,x) = \beta(x) \)

It’s not hard to see \( v_\epsilon \) converges in \( L^\infty([0,T] \times \mathbb{R}^n - *) \) to some function \( v \), satisfying the following PDE in the distribution sense

\[
\partial u \frac{\partial}{\partial t} + b \cdot \nabla u = 0 \tag{1.5}
\]

with initial data \( u(0,x) = \beta(x) \). This PDE is called the transport equation.

Observe that \( w_\epsilon := v_\epsilon^2 = \beta(X_\epsilon)^2 \) is also a solution of the PDE (1.4) with squared initial data, that is, \( w_\epsilon \) satisfies

\[
\partial w_\epsilon \frac{\partial}{\partial t} + b \cdot \nabla w_\epsilon = 0, \quad w_\epsilon(0,x) = \beta(x)^2. \tag{1.6}
\]

Clearly it’s weakly star convergent limit \( w \) satisfies the PDE (1.5) with squared initial data.

Hence to show \( v_\epsilon \) converges in measure, it suffices to show the uniqueness of the PDE (1.5) with given \( L^\infty \) data \( u_0 \). Indeed, if the uniqueness is true, then \( v^2 = w \), and hence we obtain \( v_\epsilon \) converges to \( v \) and \( v_\epsilon^2 \) converge to \( v^2 \), in \( L^\infty([0,T] \times \mathbb{R}^n - *) \), and therefore \( v_\epsilon \) converges in \( L^2([0,T];L^2_{loc}) \) to \( v \), therefore in measure.

Hence the core part is to show the uniqueness of (1.5) under \( L^\infty \) initial data.

Let me here explain heuristically how the uniqueness of the PDE (1.5) is proved. We multiply the equation (1.5) by \( 2u \), and by formally applying the chain rule we deduce

\[
\partial_t(u^2) + b \cdot \nabla(u^2) = 0. \tag{1.7}
\]

Now integrate over the space \( \mathbb{R}^d \) for every fixed value \( t \), obtaining

\[
\frac{d}{dt} \int u^2(t,x)dx = \int \text{div}(b(x))u^2(t,x)dx \leq ||\text{div}b||_{L^\infty} \int u^2(t,x)dx, \tag{1.8}
\]

thanks to the divergence theorem. Now a simple application of Gronwall’s lemma implies if the \( L^2 \) norm of the solution vanishes at the initial time, then it vanishes for all time. By linearity of (1.5) we obtain uniqueness.

In the above argument, the delicate point is hidden in the passage from (1.5) to (1.7). Indeed, since no regularity is assumed on the solution \( u \), the application of the chain rule formula is not justified. This led DiPerna and Lions [1] to define renormalized solution of (1.5) as distributional solutions \( u \) for which

\[
\partial_t[\beta(u)] + b \cdot \nabla[\beta(u)] = 0 \tag{1.9}
\]

holds in the distribution sense for any bounded \( C^1 \) function \( \beta \) with bounded derivative. In some sense, the validity of the chain rule formula is assumed by definition of the renormalized solutions. Thus if \( u \) is a renormalized solution, that is, \( u \) satisfies (1.9), then one can similarly deduce

\[
\frac{d}{dt} \int \beta(u(t,x))\phi(x)dx = \int \text{div}(b(x))\phi(x)\beta(u(t,x))dx + \int b(x)\nabla \phi(x)\beta(u(t,x)), \tag{1.10}
\]
where \( \phi \) is the standard test function. Let \( \phi \) converge to 1 and \( \beta(t) \) converge to \( |t|^p \) (\( 1 \leq p < \infty \)), one can show if the initial condition of (1.5) is zero in \( L^p \), then the solution is zero, hence the uniqueness is true if \( u_0 \in L^p \cap L^\infty \). By applying a duality argument, the same holds if \( p = \infty \).

The core part of DiPerna and Lions [1] is to show if \( u \) is a distribution solution of (1.5), then \( u \) is a renormalized solution satisfying (1.9). This is achieved via a regularization procedure combined with a control on the convergence of the error term that appears, see [1][Theorem II.1] for the exact statement and proof. Hence the core part gives well-posedness for both PDE (1.5) and ODE (1.1).

Actually, DiPerna and Lions proved in [1] the more general well-posedness result:

**Theorem 1.1.** If

\[
(1.11) \quad c, \text{div} b \in L^\infty(\mathbb{R}^N), \quad b \in L^1 \left([0, T]; W_{\text{loc}}^{1,1}(\mathbb{R}^N)\right),
\]

and

\[
(1.12) \quad \frac{b}{1 + |x|} \in L^1 \left([0, T]; L^1 + L^\infty\right),
\]

then given the initial data \( u_0 \in L^p(\mathbb{R}^N) \) (\( 1 \leq p \leq \infty \)), there exists a unique (renormalized) solution \( u \) in the space \( L^\infty([0, T]; L^p(\mathbb{R}^n)) \) of the transport equation

\[
(1.13) \quad u_t - b \nabla u + cu = 0.
\]

Moreover, if letting \( b_n, c_n \in L^1([0, T]; L^1_{\text{loc}}) \) be such that \( \text{div} b_n \in L^1([0, T]; L^1_{\text{loc}}) \) and \( b_n, c_n, \text{div} b_n \) converge to \( b, \text{div} b \) (resp.) in \( L^1([0, T]; L^1_{\text{loc}}) \). Let \( u_n \) be a bounded sequence in \( L^\infty([0, T]; L^1_{\text{loc}}) \) such that \( u^n \) is a (renormalized) solution of (1.13) with \( b, c \) replaced by \( b_n, c_n \) corresponding to an initial condition \( u^n_0 \in L^p_{\text{loc}} \). Assume that \( u^n_0 \) converges in \( L^p_{\text{loc}} \) to some \( u^0 \), then \( u_n \) converges in \( C([0, T]; L^p_{\text{loc}}) \) to the (renormalized) solution \( u \) of (1.5).

Theorem 1.1 implies the following two theorems for the well-posedness of ODE. For simplicity, we only state the autonomous case.

**Theorem 1.2.** Assuming (1.11) and (1.12) in the autonomous case, then for almost every \( x \), there exists a unique \( X \in C^1(\mathbb{R}) \) satisfying (1.1) and the following conditions:

\[
(1.14) \quad X(t + s, x) = X(t, X(s, x))
\]

\[
(1.15) \quad \lambda \circ X(t) \leq C \lambda \quad \text{for all} \; t \in \mathbb{R},
\]

where \( \lambda \) is the Lebesgue measure, \( \lambda \circ X(t) \) is the push forward of \( \lambda \) by \( X(t) \) and \( C \) is a positive constant, which is the so-called compressibility constant of the flow \( X \).

**Theorem 1.3.** Let \( b_n \in L^1_{\text{loc}} \) be such that \( \text{div} b_n \in L^1_{\text{loc}} \) and \( b_n, \text{div} b_n \) converge as \( n \) goes to \( b, \text{div} b \) in \( L^1_{\text{loc}} \) (respectively) where \( b \) satisfies (1.11) and (1.12) in the autonomous case. Assume that there exists \( X_n \) solving the ODE

\[
\frac{dX}{dt} = b_n(X)
\]

and satisfying the properties of the solution stated in Theorem 1.2, then for all \( T \in (0, \infty) \), \( X_n \) converges in \( C([0, T]; L^1 N) \) to the mapping \( X \in AC(\mathbb{R}; L^1 N) \) satisfying (1.15) and (1.14).

**Remark 1.4.** The assumption (1.15) is natural, because if \( X \) is the solution of the ODE (1.1) with \( \text{div} b \in L^\infty \), then by chain rule formally we have

\[
\frac{d}{dt} \nabla X = \nabla b(X) \nabla X, \quad \nabla X(0) = I
\]

and hence

\[
\frac{d}{dt} JX = \text{div} b JX, \quad JX(0) = 1.
\]

Therefore for any \( 0 \leq t \leq T \) by the Gronwall inequality \( JX(t) \leq C \), which implies (1.15).
We remark that all these well-posedness results of ODE can be put into the general theory of well-posedness of the continuity equation:

\[
\frac{d}{dt} \mu_x + D_x \cdot (b \mu_x) = 0 \quad (t, x) \in I \times \mathbb{R}^N.
\]

The existence and uniqueness of the measure valued solution of the PDE 1.16 for any initial data which is a finite measure, implies by the superposition principle, the existence of the ODE in the sense that each trajectory is absolutely continuous and (1.15) is satisfied, and the uniqueness of the ODE in the sense that for almost every initial data \(x\), each trajectory starting at \(x\) is unique. For a more precise statement, see [3][Section 3]. The idea of proving the superposition principle is using the weak convergence of measures, i.e. the convergence with respect to the duality with continuous and bounded functions, and the easy implication in Prokhorov compactness theorem: any tight and bounded family in \(\mathcal{M}_+\) is relatively compact w.r.t. the narrow convergence.

2. A NEW APPROACH

The arguments of the DiPerna-Lions theory are quite indirect and they exploit (via the theory of characteristics) the connection between (1.1) and the Cauchy problem for the transport equation (1.13).

I later learned that in 2008, Crippa and De Lellis in [7] recover a lot of the ODE results of the DiPerna-Lions theory from simple apriori estimates, directly in the Lagrangian formulation, under various relaxed hypotheses. Assuming the existence of a regular Lagrangian flow \(X\), they give estimates of integral quantities depending on \(X(t, x) - X(t, y)\). These estimates depend only on \(||b||_{W^{1,p}} + ||b||_\infty\) and the compressibility constant \(C\) in (1.15). Moreover, a similar estimate can be derived for the difference \(X(t, x) - X'(t, x)\) of regular Lagrangian flows of different vector fields \(b\) and \(b'\), depending only on the compressibility constant of \(b\) and on \(||b||_{W^{1,p}} + ||b||_\infty + ||b'||_\infty + ||b - b'||_{L^1}\). As direct corollaries of their estimates they then derive:

(a) Existence, uniqueness, stability, and compactness of regular Lagrangian flows;
(b) Some mild regularity properties, like the approximate differentiability proved in [5], that we recover in a new quantitative fashion.

In [7], the starting point is to derive the "local" Lipschitz estimates. The following theorem states the simplified qualitative version:

**Theorem 2.1.** Let \(p > 1\) and let \(b_i(x) = b(t, x) \in W^{1,p}_\text{loc}\) be the uniform bounded vector fields with bounded divergence, with

\[
\int_0^T \int_{B_R} |\nabla b_i|^p dx dt < \infty
\]

for all \(R > 0\). Then for every \(\epsilon, R > 0\), we can find a compact set \(K \subset B_R(0)\) such that \(|B_R(0) \setminus K| \leq \epsilon\) and the restriction of \(X\) to \([0, T] \times K\) is Lipschitz continuous.

This result is remarkable, because it shows we can recover somehow the standard Cauchy-Lipschitz theory provided we remove sets of small measure.

Their idea of proving the theorem is to estimate the local superemum of the \(\log\) of the difference quotient of \(X\) in space variable. By using some standard estimate in singular integral theory, that is, the difference quotient of \(X\) in the space variable can be bounded by the maximal function, and the \(L^p\) norm of maximal function is bounded by the \(L^p\) norm of the function itself, the authors show the \(L^p\) norm of the superemum of logarithmic term is bounded by \(L^p\) norm of \(Db\), and then by a standard measure theory argument the measure theoretical "local" Lipschitz property is deduced. Thanks to the "local" Lipschitz property for the space variable of the solution, by an Ascoli-type argument the compactness of the flows can be derived, which implies existence of ODE. For uniqueness and stability results, the maximal function still plays a major role. For the precise statement and proof see [7][Theorem 2.9].
Actually the well-posedness result is true for $Db \in L^{\log L}$, which relies on the Lipschitz estimate under the condition $Db \in L^{\log L}$. More precisely, they have the following theorem:

**Theorem 2.2.** There exist Borel maps $L_t : \mathbb{R}^N \to M^{d \times d}$ satisfying

$$\lim_{h \to 0} \frac{X(t, x + h) - X(t, x) - L_t(x)h}{h} = 0 \text{ locally in measure for any } t \in [0, T].$$

If, in addition, we assume that

$$\int_{B_R} |\nabla b| \ln(2 + |\nabla b|) dx < \infty \quad \forall R > 0$$

then the flow of ODE (1.1) has the following "local" Lipschitz property: for any $\epsilon > 0$ there exists a Borel set $A$ with

The proof is similar. The mere difference is, instead of using the $L^p$ estimate for maximal function, they use the fact that the $L^1$ norm of maximal function can be bounded by the $L^{\log L}$ norm of the function.

Although in [7] the authors proved some new results, for example the differentiability in space variable, which implies the well-posedness, they didn’t fully recover the DiPerna-Lions theory under the original assumption on $b$.

I’m really puzzled if one can recover DiPerna-Lions theory for $Db \in L^1$. A more general problem relates to the Bressan’s compactness conjecture:

Let $b_k$ be smooth vector fields and $X_k(t, x)$ solve the ODE

$$(2.1) \quad \frac{d}{dt} X_k(t, x) = b_k(t, X_k(t, x)), \quad X_k(0, x) = x$$

Assume that $||b_k||_{L^\infty} + ||\nabla b_k||_{L^1}$ is uniformly bounded and that $X_k$ satisfy (1.15) universally for $k$, then the sequence $\{X_k\}$ is strongly precompact in $L^{1}_{loca}$.

My question is whether this conjecture is true for a certain class of $b_k$, for example $b_k = b * \rho_{x_k}$, or true for a certain $\rho$, inspired by Ambrosio’s method in [2]. Indeed, Ambrosio showed by anistropic estimate that the key error term can be estimated by the form

$$\Lambda(A, \rho) := \int_{\mathbb{R}^N} | < A z, \nabla \rho(z) > | dz,$$

and then by a deep result due to Alberti,

$$\inf \left\{ \Lambda(A, \rho) : \rho \in C^\infty_c(B_1), \rho \geq 0, \int_{\mathbb{R}^N} \rho = 1 \right\} = |trace A|.$$
3. Further studies

This section is copied from Ambrosio’ survey in 2011, see [9].

(vector fields in $LD$) It was noticed in [8] that the isotropic smoothing scheme of [1], on which the uniqueness proof relies, works under the ony assumption that the symmetric part $Du + tDu$ of the distributional derivative is absolutely continuous. This vector space, usually denoted by $LD$ in the theory of linear elasticity, can be strictly larger than $W^{1,1}$. Notice however that $Du + tDu \in L^p_{loc}$ for some $p > 1$ implies $u \in W^{1,p}_{loc}$ by a local version of Korn’s inequality.

(vector fields in $BV$) Later on in 2004, Ambrosio [2] extended these results to the case of vector fields with bounded variation with respect to the space variable, i.e., the distributional derivative is a locally finite measure, and with bounded divergence. In both results, the need of controlling the spatial divergence of $b$ essentially comes from computations analogue to (1.8).

We remark that the possibility of extending the uniqueness (without assuming 1.15) result by removing the assumption $\nabla b \in L^\infty$ is ruled out by the counter example made by Beck cited in [1][Section IV]. However if the solution of the ODE satisfy (1.15), then according to [7] we do have the uniqueness for $b \in W^{1,p}, p \geq 1$ with $\nabla b$ not necessarily bounded. Also, uniqueness fails even for divergence free vector fields without integrable first derivatives, see the counter example provided in [1].

The hard point of the proof of [2] is to control the convergence of the error term in the regularization procedure in this weaker context. The argument is based on more refined arguments of geometric measure theory, in particular on some fine properties of $BV$ functions. For a general survey on this topic, see [3], [4] and [5].

(vector fields in $BD$) Recall that $BD$ consists in the space of functions $u$ such that $(Du + tDu)$ are representable by measures. The extension to $BD$ vector fields is still an open problem: indeed, one is tempted to use symmetric mollifiers as in [8], but we know that even for $BV$ vector fields anisotropic mollifiers are needed to get this result.

(vector fields representable as singular integrals) More recently Bouchut and Crippa achieved in [6] a very nice extension of the theory to vector fields $b$ that can be represented as a singular integral

$$b(x) = \int K(x - y)F(y)dy$$

with $F \in [L^1(\mathbb{R}^N)]^N$. Here $K$ is a matrix-valued map whose components satisfy the standard assumption of the theory of singular integral operators, so that the weak $L^1$ estimates are available. Notice that the class (3.1) includes $W^{1,1}$ functions $g$ (and vector fields) because the solution to $\Delta w = \nabla F$ is representable by (with appropriate boundary conditions)

$$w(x) = C(N)\int |y - x|^{2-N} \nabla F(y)dy = C(N)(N - 2)\int \frac{y - x}{|y - x|^N} F(y)dy.$$ 

Choosing $F = \nabla g$ gives $w = g$ and hence the desired representation of $g$. It is also easily seen that the class of vector fields in (3.1) is not contained in $W^{1,1}$ or in $BV$, so that definitely [6] provides new results (and also new applications to stability of solutions to incompressible Euler equations with vorticity in $L^1$). The open problem is to have an extension of this result to the case when $F$ is just a measure, and not necessarily $L^1$ function.

To conclude, we mention that there are really a lot more other topics, applications and open problems from the transport equation theory and ODE theory associated with weakly differentiable vector fields, both in Euclidean space and the general metric spaces.
4. References

References


