Algebraicity of the Metric Tangent Cones and Equivariant K-stability

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Abstract

We prove two new results on the K-polystability of $\mathbb{Q}$-Fano varieties based on purely algebro-geometric arguments. The first one says that any K-semistable log Fano cone has a special degeneration to a uniquely determined K-polystable log Fano cone. As a corollary, we combine it with the differential-geometric results to complete the proof of Donaldson-Sun's Conjecture which says that the metric tangent cone of any close point appearing on a Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds only depends on the algebraic structure of the singularity. The second result says that for any log Fano variety with a torus action, the K-polystability is equivalent to the equivariant K-polystability, that is, to check K-polystability, it is sufficient to check special test configurations which are equivariant under the torus action.

Contents

1 Introduction

We work over the field $\mathbb{C}$ of complex numbers. This paper is a sequel to the works in [Li17b, LX16, LX17]. Together with the previous works, we complete the proof of Donaldson-Sun's Conjecture [DS17, Conjecture 3.22] (see Theorem 1.1), which says that as an affine variety, the metric tangent cone $C := C_o(M_\infty, d_\infty)$ of any point o on a Gromov-Hausdorff (GH) limit (M_∞, d_∞) of a sequence of Kähler-Einstein Fano manifolds only depends on the algebraic structure of the singularity and is independent of the metric structure. Previously in [LX17] we proved that the intermediate construction W in Donaldson-Sun's work (see [DS17]) only depends on the algebraic structure.

Our strategy is to systematically use minimizers of the normalized volume functional (defined in [Li18]) to characterize valuations associated to metric tangent cones. Aiming at a vast generalization of the original differential geometric approach, we try to algebraize the construction of [DS17] by giving a completely local definition of a two-step degeneration process for an arbitrary klt singularity. This has been done under suitable assumptions about the minimizer of the normalized volume. In fact, these assumptions are posted to make the first step of the degeneration possible and our current note draws a complete picture of the second step. In particular, with the help of the metric structures to verify the assumptions, we now have a rather satisfactory understanding of this process for those singularities appearing on the GH-limit M_∞. In below, we will give more details.

1.1 Main results

For the first step of the degeneration, in [LX17], we showed that the valuation considered in [DS17], whose original definition depends on the metric, is a minimizer of the normalized volume and such a minimizer is *uniquely* determined by the underlying algebraic structure. This allows us to recover W using the minimizing valuation and hence gives the first part of [DS17, Conjecture 3.22].

Recall that the cone W degenerates to the metric tangent cone C and is K-semistable (see [DS17] and [LX17, Theorem 5.5]). In the current paper, we complete the picture by showing that the metric tangent cone C is the unique K-polystable degeneration of W. In particular, this implies that C depends only on the algebraic structure of W, which itself only depends on the algebraic structure of $o \in M_\infty$.

Theorem 1.1 ([DS17, Conjecture 3.22])**.** *The metric tangent cone C of $o \in M_\infty$ on the GH-limit of Kähler-Einstein Fano manifolds depends only on the algebraic structure of $o \in M_\infty$.*

As in [DS17], the assumption of M_∞ can be weakened, e.g. M_∞ is a GH-limit of a sequence of projective manifolds X with fixed volumes, bounded Ricci curvature and diameter. All argument verbatim extends. One can expect Theorem 1.1 will significantly simplify the determination of metric tangent cones in examples (see e.g. [HS17]).

Since a Fano cone singularity (C, ξ) with a Ricci-flat Kähler cone metric is aways K-polystable (see [CS15, Theorem 7.1] and also Corollary A.3), once knowing that W depends only on the algebraic structure of $o \in M_\infty$, Theorem 1.1 is a consequence of the following more general result by letting $(X, D, \xi_0) = (W, \emptyset, \xi_0)$:

Theorem 1.2 (Existence and uniqueness of K-polystable degenerations: log Fano cones)**.** *Given a K-semistable log Fano cone singularity (X, D, ξ_0), there always exists a special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ that degenerates (X, D, ξ_0) to a K-polystable log Fano cone singularity (X_0, D_0, ξ_0). Furthermore, such (X_0, D_0, ξ_0) is uniquely determined by (X, D, ξ_0) up to isomorphism.*

If we restrict ourselves to the quasi-regular case of log Fano cones, then we obtain the following result for log Fano varieties.

Theorem 1.3 (Existence and uniqueness of K-polystable degenerations: log Fano varieties)**.** *Given a K-semistable log Fano variety (S, B), there always exists a special test configuration $(\mathcal{S}, \mathcal{B})$ that degenerates (S, B) to a K-polystable pair (S_0, B_0). Furthermore, such (S_0, B_0) is uniquely determined by (S, B) up to isomorphism.*

We note that for the special case of $\mathbb{Q}$-Gorenstein smoothable Fano varieties, this was proved in [LWX14, 7.1] based on an analytic results on the existence and uniqueness of Gromov-Hausdorff limit for a flat family of Fano Kähler-Einstein manifolds (see also [SSY16]). Our current proof of Theorem 1.2 uses only algebro-geometric arguments and thus can remove the 'smoothable' assumption. Moreover, our techniques also give rise to an equivariant criterion for testing K-polystability.

Theorem 1.4 (T-equivariant K-stability=K-stability)**.** *Let (S, B) be a log Fano variety with an action by a torus group $T \cong (\mathbb{C}^*)^d$. Then (S, B) is K-polystable if and only if it is T-equivariantly K-polystable, that is for all T-equivariant special test configuration $(\mathcal{S}, \mathcal{B})$, the generalized Futaki invariant $\mathrm{Fut}(\mathcal{S}, \mathcal{B}) \geq 0$, and the equality holds only when the test configuration is a product, i.e. $(\mathcal{S}, \mathcal{B}) \cong (S, B) \times \mathbb{A}^1$.*

Note that Theorem 1.4 is proved for smooth Fano manifolds in [DS16] for general reductive group actions using analytic approach. Our result works for any singular $\mathbb{Q}$-Fano varieties. This combined the work [IS17] allows one to effectively check the K-stability of $\mathbb{Q}$-Fano T-varieties of complexity one.

1.2 Sketch of the proof

Now we briefly outline our proof of Theorem 1.2 and 1.3. We will first concentrate on the case of log Fano varieties, corresponding to the quasi-regular case of log Fano cones. Let $(\mathcal{S}^{(i)}, \mathcal{B}^{(i)})(i = 1, 2)$ be two special test configurations of the log Fano variety (S, B) with the central fiber $(S_0^{(i)}, B_0^{(i)})$. The main technical result Theorem 3.2 easily implies Theorem

1.3, says that if $\mathrm{Fut}(\mathcal{S}^{(i)},\mathcal{B}^{(i)})=0,(i=1,2)$, then there exist special test configurations $(\mathcal{S}'^{(i)},\mathcal{B}'^{(i)})$ of $(S_0^{(i)},B_0^{(i)})$ such that $(\mathcal{S}'^{(i)},\mathcal{B}'^{(i)})$ have isomorphic central fibers.

Similar to [Li17b, LL19, LX16], we consider the normalized volume function $\widehat{\mathrm{vol}}_{(X,D,x)}(\cdot)$ defined on the valuation space $\mathrm{Val}_{X,x}$ over the vertex x of the cone $(X,D)=C(S,B;-\lambda(K_S+B))$ for a sufficiently divisible $\lambda>0$ (see Section 2.1 for the definitions of $\widehat{\mathrm{vol}}_{(X,D,x)}(\cdot)$ and $\mathrm{Val}_{X,x}$). Then $(\mathcal{S}^{(1)},\mathcal{B}^{(1)})$ determines a "ray" of valuations, temporarily denoted by $\{w_\epsilon\}_{0\le\epsilon\ll 1}$, emanating from the canonical valuation $w_0=\mathrm{ord}_S$ (S also denotes the divisor obtained by blowing up the vertex) and we know that the generalized Futaki invariant $\mathrm{Fut}(\mathcal{S}^{(1)},\mathcal{B}^{(1)})$ is the derivative of the normalized volume at w_0 along this ray. When $k\gg 1$, $w_{1/k}=a_k\cdot\mathrm{ord}_{E_k}$, where $a_k>0$ and E_k is a Kollár component over (X,D,x) (see Definition 2.6).

By taking cones similar as before, $\{(\mathcal{S}^{(i)},\mathcal{B}^{(i)})\}_{i=1,2}$ induce special degenerations of (X,D), which will be denoted by $\{(\mathcal{X}^{(i)},\mathcal{D}^{(i)})\}_{i=1,2}$. Our key observation is that E_k can be degenerated along $(\mathcal{X}^{(2)},\mathcal{D}^{(2)})$ to obtain a model $\mathcal{Y}_k^{(2)}\to\mathcal{X}^{(2)}$ with a unique exceptional divisor $\mathcal{E}_k^{(2)}$ satisfying $(\mathcal{Y}_k^{(2)},\mathcal{E}_k^{(2)})\times_{\mathbb{C}}\mathbb{C}^*\cong(Y_k,E_k)\times\mathbb{C}^*$ where the isomorphism is compatible with the equivariant isomorphism of the *second* special test configuration. Note that $E_k\times\mathbb{C}^*$ determines a divisorial valuation over $X\times\mathbb{C}^*$ and hence over $(\mathcal{X}^{(2)},\mathcal{D}^{(2)})$. So the goal is to show that this divisorial valuation can be extracted as the only exceptional divisor over $\mathcal{X}^{(2)}$. Based on the results from the minimal model program (MMP) (see [BCHM10]), this would be true if we could find a graded sequence of ideals $\mathfrak{A}_\bullet$ and a positive real number c'_k such that the following two conditions are simultaneously satisfied:

$$(\mathcal{X}^{(2)},\mathcal{D}^{(2)}+c'_k\mathfrak{A}_\bullet)\text{ is klt}\quad\text{and}\quad A(E_k\times\mathbb{C};\mathcal{X}^{(2)},\mathcal{D}^{(2)}+c'_k\mathfrak{A}_\bullet)<1,\qquad (✿)$$

where $A(E_k\times\mathbb{C};\mathcal{X}^{(2)},\mathcal{D}^{(2)}+c'_k\mathfrak{A}_\bullet)$ is the log discrepancy of (the birational transform of) $E_k\times\mathbb{C}$ with respect to the triple $(\mathcal{X}^{(2)},\mathcal{D}^{(2)}+c'_k\mathfrak{A}_\bullet)$. Note that this way of applying MMP is also a major ingredient in the study of some related problems in [Blu18, LX16, LX17] .

To construct such a graded sequence $\mathfrak{A}_\bullet$ of ideals, we look at the graded sequence of valuative ideals $\{\mathfrak{a}_\bullet\}$ of ord_{E_k} and its equivariant degeneration along the second special test configuration $(\mathcal{X}^{(2)},\mathcal{D}^{(2)})$. The resulting graded sequence of ideals over $\mathcal{X}^{(2)}$ will be denoted by $\mathfrak{A}_\bullet$, which we claim is exactly what we are looking for. Indeed, as we will show (see Claim 3.6), the assumptions that (S,B) is K-semistable and $\mathrm{Fut}(\mathcal{S}^{(1)},\mathcal{B}^{(1)})=0$ guarantee the existence of c'_k satisfying the two conditions in (✿). This is possible thanks to the interaction between K-semistability and minimization of normalized volumes/normalized multiplicities.

Applying the relative Rees algebra construction to $\mathcal{E}_k^{(2)}\subset\mathcal{Y}_k^{(2)}/\mathbb{C}$, and then taking a quotient by the natural rescaling $\mathbb{C}^*$-action, one can obtain a family over $\mathbb{C}^2$, whose restriction to $\mathbb{C}\times\{t\}$ for $t\neq 0$ is the same as $(\mathcal{S}^{(1)},\mathcal{B}^{(1)})$ and it gives a degeneration of $(S_0^{(1)},B_0^{(1)})$ when restricted to $\mathbb{C}\times\{0\}$. On the other hand, over $\{0\}\times\mathbb{C}$, one get a degeneration of $(S_0^{(2)},B_0^{(2)})$. Therefore, we obtain that the two log Fano varieties $(S_0^{(i)},B_0^{(i)})$ $(i=1,2)$, which are special fibers of the two special test configurations $(\mathcal{S}^{(i)},\mathcal{B}^{(i)})$ $(i=1,2)$ with $\mathrm{Fut}(\mathcal{S}^{(i)},\mathcal{B}^{(i)})=0$ $(i=1,2)$, indeed admit degenerations with *isomorphic* special fibers (see Theorem 3.2).

To confirm Donaldson-Sun's Conjecture (see [DS17, Conjecture 3.22]), it is necessary to also treat the case of a general log Fano cone, i.e. including the irregular case. We first establish the condition (✿) using an approximation argument to get common degenerations as in the quasi-regular case. However, the common degenerations are a priori only *weakly special* (Definition 2.17). So we extend [LX14, Theorem 4] to (possibly irregular) log Fano cones. In other words, we reproduce the last step of [LX14] for a log Fano cone (see Section 4.2). We also apply [CS15] to obtain the K-polystability of a Ricci-flat Kähler cone singularity.

We now outline the organization of the paper. More details will be given at the beginning of each section. In Section 2.1, we recall some basic tools needed in our arguments including normalized volumes, normalized multiplicities and Kollár components. In Section 2.2, we recall the notions of log Fano cones, their test configurations and K-stability. We also discuss how to construct test configurations using models over log Fano cones. In the quasi-regular case, we are reduced to the K-stability of log Fano pairs. In Section 3, we prove our main results in the case of log Fano pairs. In Section 3.1, we prove a lemma about special degenerations of K-semistable log Fano pairs with zero Futaki invariants. In Section 3.2, we prove the main technical result (Theorem 3.2) on the existence of a common special degenerations of special degenerations with zero Futaki invariants. In Section 3.3, we finish the proof of main results for log Fano pairs. In Section 4, we deal with the general case of log Fano cones. In Section 4.1, we obtain common weakly special degenerations for log Fano cones with vanishing generalized Futaki invariants. In Section 4.2, we show that these weakly special test configurations are indeed special test configurations. we generalize the last step of results in [LX16] to the case of log Fano cones. We complete the proof of Theorem 1.2 and Donaldson-Sun's conjecuture in Section 4.3. In Appendix A, we prove the analytic result that Ricci-flat Kähler cones are Ding-polystable among $\mathbb{Q}$-Gorenstein test configurations. This result could substitute results in Section 4.2 to complete the proof of Theorem 1.1.

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Notation and Conventiones: We follow the standard notation in [KM98, Kol13]. We call a pair (S, B) a *log Fano* variety if (S, B) has klt singularities, and $-(K_S + B)$ is ample.

We will use the following terminology introduced in [Kol18]. We consider a $f\colon (X, D) \to C$ where X is normal flat over a smooth projective curve C, D is an effective $\mathbb{Q}$-divisor on X. We say (X, D) is *locally stable over* C if $(X, D + X_t)$ is log canonical for any $t \in C$.

2 Preliminaries

2.1 Normalized volumes

In this section, we recall the definition of the normalized volume of valuations centered at a klt singularity $x \in (X, D)$. This is introduced in [Li18]. For readers' convenience, in below we discuss some basic properties which will be needed later.

Definition 2.1. *Let $X = \mathrm{Spec}_{\mathbb{C}}(R)$ be an affine variety and $x \in X$ be a closed point. We denote by $\mathrm{Val}_{X,x}$ the space of real valuations $v : R \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ that satisfy the following conditions: for any $f, g \in R$:*
(1) $v(fg) = v(f) + v(g)$; (2) $v(f + g) \geq \min\{v(f), v(g)\}$; (3) $v(0) = +\infty, v(\mathbb{C}^*) = 0$; (4) $v(f) > 0$ if $f(x) = 0$.

For any $v \in \mathrm{Val}_{X,x}$ and $m \in \mathbb{R}$, its valuative ideal (at level m) is defined as $\mathfrak{a}_m(v) := \mathfrak{a}_m(v, X) = \{f \in R;\ v(f) \geq m\}$.

We remark that $\mathrm{Val}_{X,x}$ is also called the 'non-archimedean link' around $x \in X$ in some literatures.

For any $m > 0$, $\mathfrak{a}_m(v)$ is a primary ideal associated to the maximal ideal $\mathfrak{m}_x$. We will denote its Hilbert-Samuel multiplicity by $\mathrm{mult}(\mathfrak{a}_m)$. If $\Lambda = v(R) \subset \mathbb{R}_{\geq 0}$ denotes the valuative semigroup of v, then $\{\mathfrak{a}_m(v);\ m \in \Lambda\}$ is a Λ-graded sequence of ideals. In other words, they satisfy, for any $m, m' \in \Gamma$, $(i) : \mathfrak{a}_{m'}(v) \subseteq \mathfrak{a}_m(v)$ if $m' \geq m$ and $(ii) : \mathfrak{a}_m(v) \cdot \mathfrak{a}_{m'}(v) \subseteq \mathfrak{a}_{m+m'}(v)$. Note that $\{\mathfrak{a}_m(v); m \in \mathbb{Z}\}$ is also a $\mathbb{Z}$-graded sequence of ideals.

Definition-Proposition 2.2 ([ELS03, Mus02])**.** *For any $v \in \mathrm{Val}_{X,x}$, the volume of v is the following well-defined quantity:*

$$\mathrm{vol}(v) = \lim_{m\to+\infty} \frac{\dim_{\mathbb{C}}(R/\mathfrak{a}_m(v))}{m^n/n!} = \lim_{m\to+\infty} \frac{\mathrm{mult}(\mathfrak{a}_m)}{m^n} =: \mathrm{mult}(\mathfrak{a}_\bullet). \tag{1}$$

Now we assume (X, D) is a log pair such that $K_X + D$ is $\mathbb{Q}$-Cartier. For any divisorial valuation $v = \mathrm{ord}_S$ where S is a divisor of Y for a birational morphism $\mu : Y \to X$, the *log discrepancy* of ord_S is defined as $A_{(X,D)}(\mathrm{ord}_S) = \mathrm{ord}_S(K_Y - \mu^*(K_X + D)) + 1$. By [JM12] and [BFFU15], there is a canonical way to extend the log discrepancy to become a lower semicontinuous function $A_{(X,D)} : \mathrm{Val}_{X,x} \to \mathbb{R} \cup \{+\infty\}$.

Definition-Proposition 2.3 (see [Li18, Theorem 1.1]). *Assume $x \in (X, D)$ is a klt singularity. For any $v \in \mathrm{Val}_{X,x}$, its normalized volume $\widehat{\mathrm{vol}}_{(X,D,x)}(v)$ is defined as:*

$$\widehat{\mathrm{vol}}_{(X,D,x)}(v) = \begin{cases} A_{(X,D)}(v)^n \cdot \mathrm{vol}(v), & \text{if} \quad A_{(X,D)}(v) < +\infty; \\ +\infty, & \text{if} \quad A_{(X,D)}(v) = +\infty. \end{cases} \tag{2}$$

For simplicity, we will just write $\widehat{\mathrm{vol}}(v)$ if the singularity $x \in (X, D)$ is clear. This quantity is a rescaling invariant: $\widehat{\mathrm{vol}}(\lambda v) = \widehat{\mathrm{vol}}(v)$ for any $\lambda > 0$.

The volume of a klt singularity $x \in (X, D)$ is defined to be the following positive number

$$\mathrm{vol}(x, X, D) = \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_{(X,D,x)}(v). \tag{3}$$

It has been shown that there always exists a minimizer v of $\widehat{\mathrm{vol}}_{(X,D,x)}$ among all $v \in \mathrm{Val}_{X,x}$ in [Blu18]. The expected properties of the minimizers are formulated in the *Stable Degeneration Conjecture* ([Li18, Conjecture 6.1], [LX17, Conjecture 1.2]). The case of cone singularities over Fano varieties was studied in [Li17b, LL19]. The general case was systematically studied in [LX16] under the assumption that the minimizer is a divisorial valuation and in [LX17] under the assumption that the minimizer is a higher rank quasi-monomial valuation.

We will need a relation between the normalized volume and the normalized multiplicity of a graded sequence of ideals.

Proposition 2.4 ([Liu18]). *If $x \in (X, D)$ is an n-dimensional klt singularity, then we have*

$$\mathrm{vol}(x, X, D) = \inf_{\mathfrak{b}_\bullet} \mathrm{mult}(\mathfrak{b}_\bullet) \cdot \mathrm{lct}^n(X, D; \mathfrak{b}_\bullet),$$

where $\mathfrak{b}_\bullet$ runs over all graded sequence of primary ideals cosupported at x.

We now state some central results from our previous works and refer to the next section for the notations of log Fano cones (see Definition 2.12) and their K-stability (see e.g. Definition 2.23).

Theorem 2.5 ([Li17b, LL19, LX16, LX17]). *Let (X, D, ξ) be a log Fano cone singularity. Then it is K-semistable if and only if the valuation wt_ξ induced by ξ is a minimizer of $\widehat{\mathrm{vol}}_{(X,D,x)}$ on $\mathrm{Val}_{X,x}$.*

We will also use the following notion frequently:

Definition 2.6. *Let (X, D, x) be a klt singularity. A divisor S over (X, D, x) is called a Kollár component over (X, D, x), if there exists a birational morphism $\mu : Y \to X$ such that (i) μ is an isomorphism over $X \setminus \{x\}$ and the exceptional divisor $S = \mu^{-1}(x)$ is irreducible and $\mathbb{Q}$-Cartier; (ii) $-S$ is μ-ample; (iii) $(Y, \mu_*^{-1}D + S)$ is plt.*

The relevance of Kollár components to the minimization of normalized volume is contained in the following result:

Theorem 2.7 ([LX16]). *Let (X, D, x) be a klt singularity. Assume that $v_0 \in \mathrm{Val}_{X,x}$ is a minimizer of $\widehat{\mathrm{vol}}_{(X,D,x)}$. Then we can find a sequence of Kollár components S_k and a constant $c_k > 0$, such that*

$$c_k \cdot \mathrm{ord}_{S_k} \to v_0 \quad \text{and} \quad \widehat{\mathrm{vol}}(\mathrm{ord}_{S_k}) \to \widehat{\mathrm{vol}}(v_0) \ \text{as } k \to +\infty.$$

Moreover, if v_0 is divisorial, then v_0 is given by a K-semistable Kollár component.

Proof. See [LX16, Theorem 1.2 and Theorem 1.3]. □

In the above theorem, when v_0 is a divisorial minimizer, then [Blu18] also shows it yields a Kollár component. In the case of K-semistable log Fano cones, the approximation stated in the above theorem can be realized concretely by perturbing the Reeb vector field to rational ones.

2.2 K-stability of log Fano cones

In this section, we recall the definition of a log Fano cone singularity and its K-stability, by essentially following [CS18, CS15] and [LX17, Section 2.5]. Denote by T a complex torus which is isomorphic to $(\mathbb{C}^*)^r$.

Test configurations

Definition 2.8. *Let X be an n-dimensional reduced affine variety which is not necessarily irreducible. We say that a T-action on X is good if it is effective and there is a closed T-fixed point $x \in X$ (called the vertex) that is in the closure of any T-orbit. By a T-singularity in this paper, we always mean an affine variety X with a good T-action. If D is a T-invariant $\mathbb{R}$-divisor on X we say that (X, D) is a pair with a good T-action.*

Let $N = \mathrm{Hom}(\mathbb{C}^*, T)$ be the co-weight lattice and $M = N^*$ the weight lattice. If $X = \mathrm{Spec}_{\mathbb{C}}(R)$ is a T-variety, then there is a weight space decomposition:

$$R = \bigoplus_{\alpha \in \Gamma} R_\alpha \text{ where } \Gamma = \{\alpha \in M \mid R_\alpha \neq 0\} \subset M. \tag{4}$$

The action being good implies $R_0 = \mathbb{C}$. We will call any element ξ in the Lie algebra $\mathfrak{t}_{\mathbb{R}} := N \otimes \mathbb{R}$ *a coweight vector* (or abbreviated as *a vector*). We will denote by $\langle \xi \rangle$ the sub-torus of T generated by ξ, i.e. the sub-torus corresponds to the minimal linear $\mathbb{Q}$-linear subspace $V \subset N \otimes \mathbb{Q}$ such that $V \otimes \mathbb{R}$ contains ξ.

If T acts on a smooth variety X, then ξ will give a vector field on X. For example, if we consider the multiplication of $\mathbb{C}^*$ on $\mathbb{C}$, then the coweight vector $1 \in \mathbb{Z}$ yields the vector field $t\partial_t$.

Definition 2.9. *The Reeb cone of X with respect to a good T-action is the following set:*

$$\mathfrak{t}^+_{\mathbb{R}} := \{\xi \in N_{\mathbb{R}} \mid \langle \alpha, \xi \rangle > 0 \text{ for any } \alpha \in \Gamma \backslash \{0\}\}. \tag{5}$$

Any vector $\xi \in \mathfrak{t}^+_{\mathbb{R}}$ will be called a Reeb vector on the T-variety X.

Definition 2.10. *For any $\xi \in \mathfrak{t}^+_{\mathbb{R}}$, we define its volume as:*

$$\mathrm{vol}_X(\xi) := \mathrm{vol}_{X,x}(\xi) = \lim_{k \to \infty} \frac{\sum_{\langle \xi, \alpha \rangle \leq k} \dim_{\mathbb{C}}(R_\alpha)}{k^n/n!}.$$

One key property of the volume function is the following.

Lemma 2.11 (see [CS18, LX17]). *The function $\xi \mapsto \mathrm{vol}_{X,x}(\xi)$ is smooth and strictly convex on $\mathfrak{t}^+_{\mathbb{R}}$.*

Proof. The smoothness was proved in [CS18, Theorem 4.10] where $\mathrm{vol}_X(\xi)$ was interpreted as the leading coefficient of the expansion of the so-called index character (see [MSY08]).

The convexity of $\mathrm{vol}_{X,x}$ follows from [LX17, Section 3.2]. In fact, if we let $Y \to X$ be the normalization of X, the T-action can be lifted to Y. Denote the preimage of x to be $\{y_i\}_i$, then we know y_i are on pairwise distinct components Y_i of Y, and the T-action on each Y_i is good with

$$\mathrm{vol}_{X,x}(\xi) = \sum_i \mathrm{vol}_{Y,y_i}(\xi).$$

Thus we may assume X to be normal. Then [LX17, Proposition 3.10], which generalizes the convexity result from [MSY08], says that $\xi \mapsto \mathrm{vol}_{X,x}(\xi)$ is a strictly convex function for $\xi \in \mathfrak{t}^+_{\mathbb{R}}$.

□

If (X, x) is a normal affine T-variety, then each $\xi \in \mathfrak{t}^+_{\mathbb{R}}$ corresponds to a valuation $\mathrm{wt}_\xi \in \mathrm{Val}_{X,x}$ which is defined as:

$$\mathrm{wt}_\xi(f) = \min \left\{ \langle \alpha, \xi \rangle ; f = \sum_\alpha f_\alpha \text{ with } f_\alpha \neq 0 \in R_\alpha \right\}. \tag{6}$$

Definition 2.12 (Log Fano cone singularity). *Let (X, D) be an affine pair with a good T action. Assume (X, D) is a normal pair with klt singularities. Then for any $\xi \in \mathfrak{t}^+_{\mathbb{R}}$, we call the triple (X, D, ξ) a log Fano cone structure that is polarized by ξ. If $\langle \xi \rangle \cong \mathbb{C}^*$ which is equivalent to saying that ξ is a multiple of a vector in $\mathfrak{t}^+_{\mathbb{Q}}$, then we call (X, D, ξ) to be quasi-regular. Otherwise, we call it irregular.*

Definition 2.13 (Quotient in the quasi-regular case)**.** *In the quasi-regular case, we can take the quotient (S, B) of $(X \setminus \{x\}, D \setminus \{x\})$ by the $\mathbb{C}^*$-group $\langle\xi\rangle$ generated by ξ in the sense of a Seifert $\mathbb{C}^*$-bundle, and we will denote by $(X, D)/\langle\xi\rangle$. More precisely, assume $\xi \in \frac{1}{l}N$, and we write*

$$R = \bigoplus_{k=0} \left(\bigoplus_{\langle\xi,\alpha\rangle = k/l} R_\alpha \right) := \bigoplus_{k=0} R_k^\xi.$$

Then we take $S = \mathrm{Proj}(\bigoplus_{k=0} R_k^\xi)$. By [Kol04, Section 4], $\pi\colon X \setminus \{x\} \to S$ is a Seifert $\mathbb{C}^*$-bundle, with the quotient $X \setminus \{x\} \to (S, B_1)$ where B_1 is the branch divisor. Write $D = \sum_i a_i D_i$. Since each D_i is $\mathbb{G}_m$-invariant, D_i is the pull back of a divisor E_i on S and the multiplicity of D_i along $\pi^*(E_i)$ is denoted by m_i. Define $B_2 = \sum_i \frac{a_i}{m_i} E_i$. Let $B = B_1 + B_2$. Then $\pi^*(K_S + B) = (K_X + D)|_{X\setminus\{x\}}$ since $\pi^*(K_X + B_1) = K_{X\setminus\{x\}}$ (see [Kol04, Corollary 41]) and $\pi^*(B_1) = D|_{X\setminus\{x\}}$.

The quotient (S, B) is a log Fano variety, because we assume that (X, D) is klt at x (see [Kol04, 42] or [Kol13, Lemma 3.1]).

Definition 2.14 (Test configuration)**.** *Let (X, D, ξ_0) be a log Fano cone singularity. A T-equivariant test configuration (or simply called a test configuration) of (X, D, ξ_0) is a quadruple $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with a map $\pi : (\mathcal{X}, \mathcal{D}) \to \mathbb{C}$ satisfying the following conditions:*

1. *(1) $\pi\colon \mathcal{X} \to \mathbb{C}$ is a flat family where $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$ is normal affine (thus $\mathcal{R}$ is a finitely generated flat $\mathbb{C}[t]$-algebra), $\mathcal{D}$ is a divisor on $\mathcal{X}$ with $\mathrm{Supp}(\mathcal{D})$ not containing any component of the fiber, and there is an isomorphism $\phi\colon (\mathcal{X}, \mathcal{D}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, D) \times \mathbb{C}^*$. The torus T acts on $(\mathcal{X}, \mathcal{D}) \to \mathbb{C}$ fiberwisely, and coincides with the action on the first factor when restricted to $(\mathcal{X}, \mathcal{D}) \times_{\mathbb{C}} \mathbb{C}^* \overset{\phi}{\cong} (X, D) \times \mathbb{C}^*$. We write $\mathcal{R} = \bigoplus_\alpha \mathcal{R}_\alpha$ as decomposition into weight spaces.*
2. *(2) A $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{D})$ such that π is $\mathbb{C}^*$-equivariant where $\mathbb{C}^*$ acts on the base $\mathbb{C}$ by the multiplication and $\phi : (\mathcal{X}, \mathcal{D}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, D) \times \mathbb{C}^*$ is $\mathbb{C}^*$-equivariant. We denote the coweight vector of $\mathbb{C}^*$ by η.*
3. *(3) The torus T-action commutes with the $\mathbb{C}^*$-action.*

Condition (1) implies that each weight piece $\mathcal{R}_\alpha$ is a flat $\mathbb{C}[t]$-module. So X and X_0 have the same weight cone and Reeb cone with respect to the fiberwise T-action. In particular, ξ_0 is contained in the Reeb cone of X_0 under the T-action.

A test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is called a product one if there is a T-equivariant isomorphism $(\mathcal{X}, \mathcal{D}) \cong (X, D) \times \mathbb{C}$ and $\eta = \eta_0 + t\partial_t$ where η_0 is a coweight vector of T and $t\partial_t$ is the canonical lifting of $t\partial_t$ on $\mathbb{C}$ through the second projection. In this case, we will denote $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ by

$$(X \times \mathbb{C}, D \times \mathbb{C}, \xi_0; \eta) =: (X_{\mathbb{C}}, D_{\mathbb{C}}, \xi_0; \eta).$$

A normal test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is called $\mathbb{Q}$-Gorenstein if $K_{\mathcal{X}} + \mathcal{D}$ is $\mathbb{Q}$-Cartier.

Remark 2.15. *Our Definition 2.14, by the argument in* [LX16, Section 6]*, implies that there exists an embedding*

$$(x \in X) \subset (0 \in \mathbb{C}^N),$$

such that $\mathcal{X}$ is obtained by a one parameter group of the ambient space $0 \in \mathbb{C}^N$ which also corresponds to a weighted blow up of $\mathbb{C}^N$. The latter was used in the definition in [CS18]*. So our definition of test configurations is indeed equivalent to the one in* [CS18].

Because $K_{\mathcal{X}} + \mathcal{D}$ is $\mathbb{Q}$-Cartier, by the structure theory of T-varieties, there exists a $T \times \mathbb{C}^*$-equivariant nowhere-vanishing section $s \in |m(K_{\mathcal{X}} + \mathcal{D})|$ (see [LS13, Proposition 4.4], and also [MSY08, 2.7]). For any $b \in \mathbb{R}$ and $\xi + b\eta \in \mathfrak{t}_{\mathbb{R}} \oplus \mathbb{R}\eta \cong N_{\mathbb{R}} \oplus \mathbb{R}$, define:

$$A(\xi + b\eta) := \frac{1}{m} \frac{\mathcal{L}_{\xi+b\eta} s}{s},$$

where $\mathcal{L}_{\xi+b\eta}$ is the Lie derivative of s with respect to the vector field associated to $\xi + b\eta$. Note that this is a linear function. We remark that when $\xi + b\beta$ is in the Reeb cone, then it yields a valuation $w_{\xi+b\beta}$ and we have $A(\xi + b\eta) = A_{\mathcal{X},\mathcal{D}}(w_{\xi+b\beta})$ (see [LX17, Lemma 2.18]).

If $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is any $\mathbb{Q}$-Gorenstein test configuration of an n-dimensional log Fano cone (X, D, ξ_0), we will denote:

$$T_{\xi_0}(\eta) = \frac{A(\xi_0)\eta - A(\eta)\xi_0}{n}. \tag{7}$$

Definition 2.16 (Special test configuration)**.** *Notation as above. A special test configuration of (X, D, ξ_0) is a $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with central fiber (X_0, D_0) satisfying moreover that:*

(4) (X_0, D_0) has klt singularities.

In this case, we say that (X_0, D_0) is a special degeneration of (X, D).

We need also consider a larger class of test configurations than special ones.

Definition 2.17 (Weakly special test configuration)**.** *Notation as above. A weakly special test configuration of (X, D, ξ_0) is a $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with central fiber (X_0, D_0) satisfying moreover that:*

(5) $(\mathcal{X}, \mathcal{D} + X_0)$ has log canonical singularities.

In this case, we say that (X_0, D_0) is a weakly special degeneration of (X, D).

For simplicity, we will just say that $(\mathcal{X}, \mathcal{D})$ is a $\mathbb{Q}$-Gorenstein (or weakly special, special) test configuration if ξ_0 and η are clear. We also say that (X, D, ξ_0) degenerates to (X_0, D_0, ξ_0) (or simply to (X_0, D_0)).

Test configuration and filtration

In [BHJ17, Section 2.5], a filtration viewpoint for test configurations is developed. Here we will mainly work with data over the vertex of the cone which brings more flexibility when applying the minimal model program. In this section, we will discuss these ideas and modify them to fit into our context.

Lemma 2.18. *Given a normal T-equivariant test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ of (X, D), we can find a $\mathbb{Z}$-graded sequence of ideals $\{\mathfrak{a}_\bullet\}$ of R such that*

1. *$\mathfrak{a}_k = R$ for $k \le 0$;*
2. *$\mathfrak{a}_k$ is a homogeneous ideal for any $k \in \mathbb{Z}$: $\mathfrak{a}_k = \bigoplus_\alpha \mathfrak{a}_k \cap R_\alpha$ for any $k \in \mathbb{Z}$;*
3. *the extended Rees algebra $\mathcal{R}ees := \bigoplus_{k\in\mathbb{Z}} t^{-k}\mathfrak{a}_k$ satisfies* $\mathrm{Spec}(\mathcal{R}ees) = \mathcal{X}$.

Moreover, if η is in the Reeb cone of X_0, then $\mathfrak{a}_k$ is primary for $k > 0$.

Proof. Recall by the definition of the test configuration, $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$ where $\mathcal{R} = \bigoplus \mathcal{R}_\alpha$ and each $\mathcal{R}_\alpha$ is a flat $\mathbb{C}[t]$-module. For any $f \in \mathcal{O}_X$, we could denote by $\bar{f}$ its pull back from the first factor of $X \times \mathbb{C}^*$. Since $\mathcal{X} \times_{\mathbb{C}} \mathbb{C}^* \cong X \times \mathbb{C}^*$, we could mimic the construction in [BHJ17, Section 2.5] by defining $\mathfrak{a}_k = \bigoplus_\alpha \left\{ f \in R_\alpha \,|t^{-k}\bar{f} \in \mathcal{R}_\alpha \right\}$, and then we form the extended Rees algebra $\mathcal{R}ees = \bigoplus_{k\in\mathbb{Z}} \mathfrak{a}_k t^{-k}$.

Then it is clear by the definition we have $\mathcal{R} \cong \mathcal{R}ees$. (In particular, this implies that $\mathcal{R}ees$ is finitely generated.)

Since $\mathcal{R}$ is a flat $\mathbb{C}[t]$-algebra, that means $\mathfrak{a}_0 = R$ which implies that $\mathfrak{a}_k = R$ for $k \le 0$. This is the first property. The second property follows from that the $\mathbb{C}^*$-action generated by η commutes with T.

Finally, if η is in the Reeb cone, then $\langle \eta, \alpha \rangle > 0$ for any $\alpha \in \Gamma \backslash \{0\}$ (see (4)). Thus for any $\alpha \ne 0$ and $f \in R_\alpha$, the order of f vanishing along $(t = 0)$ is $\langle \eta, \alpha \rangle > 0$, which implies for any k, $f^m \in \mathfrak{a}_k$ for $m \gg 0$. □

Remark 2.19. *Since the Reeb cone with respect to T is open, for any given test configuration, one can always perturb ξ_0 to a rational Reeb vector and modify η accordingly so that for $m \gg 1$ sufficiently divisible, $m\xi_0 + \eta$ is an integral vector in the Reeb cone with respect to $\tilde{T} := T \times \mathbb{C}^*$.*

We give a way of obtaining test configurations using models. It generalizes the construction of special test configurations via Kollár components as discussed in [LX17, 3.1].

Definition 2.20. *Let (X, D, ξ_0) be a log Fano cone singularity. Let $\mu\colon Y \to X$ be an isomorphism outside $X \setminus \{x\}$ with a T-equivariant integral Weil divisor E supported on* $\mathrm{Ex}(\mu)$ *such that $-E$ is ample. Denote by $\mathcal{R} := \bigoplus_{k\in\mathbb{Z}} t^{-k}\mathfrak{b}_k$, where $\mathfrak{b}_k = \mu_*(\mathcal{O}_Y(-kE))$.*

Then $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is a test configuration associated to the model $\mu : Y \to X$, where $\mathcal{X} :=$ $\mathrm{Spec}(\mathcal{R})$ *and $\mathcal{D}$ is the cycle (with $\mathbb{Q}$-coefficients) degeneration of D. More precisely, if we write $D = \sum a_i D_i$, where D_i are prime divisors with the corresponding ideal I_{D_i}, then we can define $\mathcal{D}_i$ on $\mathcal{X}$ to be the divisor corresponding to the ideal $\mathcal{I}_{\mathcal{D}_i} := \bigoplus_{k\in\mathbb{Z}}(\mathfrak{b}_k \cap I_{D_i})t^{-k} \subset \mathcal{R}$, and let $\mathcal{D} = \sum a_i\mathcal{D}_i$.*

Conversely, starting with a test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ and assuming η is in the Reeb cone of X_0, we take the primary ideals $\mathfrak{a}_k$ as in Lemma 2.18, and then take the normalized filtered blow up (see [TW89, Chapter 1] for the definition) $\mu\colon Y \to X$ induced by $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k\in\mathbb{Z}}$. Then the pull back of $\mathrm{Proj}\bigoplus_{k=0}\mathfrak{a}_k/\mathfrak{a}_{k+1}$ on Y gives us the divisor E.

Lemma 2.21. *The above two constructions give equivalence between normal test configurations $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with η in the Reeb cone and models $\mu\colon Y \to X$ satisfying the conditions in Definition 2.20. Moreover,*

1. *$(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is a special test configuration if and only if $\mu\colon Y \to X$ yields a Kollár component; and*
2. *$(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is weakly special if and only if $(Y, E + \mu_*^{-1}D)$ is log canonical.*

Proof. If we start with a normal test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$, then we get a graded sequence of primary ideals $\{\mathfrak{a}_\bullet\}$ by Lemma 2.18. If we take the filtered blow up of $\{\mathfrak{a}_\bullet\}$ and get E as above, then we claim it is normal and the algebra $\{\mathfrak{b}_k = \mu_*(\mathcal{O}_Y(-kE))\}$ is the same as the algebra $\bigoplus_{k=0}\mathfrak{a}_k$.

In fact, $\bigoplus_{k=0}\mathfrak{a}_k \subset \bigoplus_{k=0}\mathfrak{b}_k$ is a subalgebra, but the latter is integral over the former. Thus it suffices to verify that the R-algebra $\bigoplus_{k=0}\mathfrak{a}_k$ is integrally closed. Similar to the proof of [Laz04, 9.6.6], this follows from the fact that to check whether a function f is contained in $\mathfrak{a}_k$ suffices to only check it at the divisorial valuation along the the special fiber X_0. More precisely, let the special fiber $X_0 = \sum m_iE_i$ where E_i are the prime divisors, then

$$\begin{aligned} &\text{a homogeneous element } f \in \text{ the normal closure } \overline{\bigoplus_{k=0}\mathfrak{a}_k} \\ \Leftrightarrow\quad & f \text{ satisfies an equation } f^m + a_1f^{m-1} + \cdots + a_m = 0 \text{ with } a_i \in \mathfrak{a}_{ik}, \end{aligned}$$

which implies the vanishing order of f along E_i is at least km_i as the element in $\mathfrak{a}_j$ have vanishing order along E_i at least jm_i by the definition. Then we conclude $f \in \mathfrak{a}_k$.

If we start with a normal model $\mu\colon Y \to X$ and E as in Definition 2.20, then $\bigoplus_{k=0}\mathfrak{b}_k$ is a normal algebra where $\mathfrak{b}_k = \mu_*(\mathcal{O}_Y(-kE))$, then we can easily show the Rees algebra $\bigoplus_{k\in\mathbb{Z}} t^{-k}\mathfrak{b}_k$ is normal, thus the induced test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is normal. If we take the filtered blow up then $Y \cong \mathrm{Proj}(\bigoplus_{k=0}\mathfrak{b}_k)$ as $-E$ is ample, and the divisor $\mathrm{Proj}(\bigoplus_{k=0}\mathfrak{b}_k/\mathfrak{b}_{k+1}) \subset Y$ yields E.

To prove the second part of the statement, let $v\colon \mathbb{A}^1_{\mathbb{C}} \subset \mathcal{X}$ corresponds to the section of vertices. Consider the $\mathbb{C}^*$-action given by η in the data of the test configuration, then $(\mathcal{S} = \mathrm{Proj}_{\mathbb{C}[t]}\mathcal{R}ees, \mathcal{B})$ is the base of the $\mathbb{C}^*$-quotient of $(\mathcal{X} \setminus v(\mathbb{A}^1_{\mathbb{C}}), \mathcal{D} \setminus v(\mathbb{A}^1_{\mathbb{C}}))$ as a Seifert bundle (see [Kol04]), i.e., we remember the codimension one orbifold structure and put it into $\mathcal{B}$. Over the special fiber, we have

$$S_0 \cong \mathrm{Proj}\bigoplus_{k=0}\mathfrak{a}_k/\mathfrak{a}_{k+1} \cong \mathrm{Proj}\bigoplus_{k=0}\mathfrak{b}_k/\mathfrak{b}_{k+1} \cong E.$$

If $(\mathcal{X}, \mathcal{D} + \mathcal{X}_0)$ is log canonical, then X_0 is reduced and (X_0, D_0) is semi-log-canonical. Thus $E = S_0$ is reduced and $(S_0, B_0 := \mathcal{B}|_{S_0})$ is semi-log-canonical. Moreover, if we write $K_E + D_E = (K_Y + E + \mu_*^{-1}D)|_E$, then D_E is sent to B_0 under the isomorphism between E and S_0. Thus by inversion of adjunction, $(Y, E + \mu_*^{-1}D)$ is log canonical.

For the converse, assume (E, D_E) is log canonical, it suffices to show that $\mathfrak{b}_k/\mathfrak{b}_{k+1} = H^0(E, \mathcal{O}_E(-kE))$ for any positive integer k, as this implies that (X_0, D_0) is the orbifold cone over (E, D_E). First, since the test configuration is $\mathbb{Q}$-Gorenstein, $-K_E - D_E \sim_{\mathbb{Q}} \lambda E|_E$ for some $\lambda > 0$. Therefore, E and $K_Y + E + \mu_*^{-1}D$ are anti-ample over X,

$$-(k+1)E = K_Y + E + \mu_*^{-1}D - (k+1)E - (K_Y + E + \mu_*^{-1}D),$$

we conclude that $R^1\mu_*\mathcal{O}_Y(-(k+1)E) = 0$ by the Kawamata-Viehweg Vanishing Theorem, then we can apply μ_* to the following exact sequence

$$0 \to \mathcal{O}_Y(-(k+1)|_E) \to \mathcal{O}_Y(-kE) \to \mathcal{O}_E((-kE)|_E) \to 0.$$

If we specialize the argument to the plt case, we obtain that E is indeed a Kollár component. $\square$

Generalized Futaki invariants and K-stability

We define the generalized Futaki invariant for $\mathbb{Q}$-Gorenstein test configuration using the volume function. One can easily show this definition is the same as the one in [CS18]. However, the formula in Definition 2.22 more fits the argument in the current paper.

Definition 2.22 (Generalized Futaki invariant)**.** *For any $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ of (X, D, ξ_0) with the central fiber (X_0, D_0, ξ_0), its generalized Futaki invariant is defined as*

$$\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) := \frac{D_{-T_{\xi_0}(\eta)}\mathrm{vol}_{X_0}(\xi_0)}{\mathrm{vol}_{X_0}(\xi_0)}.$$

Since generalized Futaki invariant defined above only depends on the data on the central fiber, we will also denote it by $\mathrm{Fut}(X_0, D_0, \xi_0; \eta)$*.*

In the above definition, we used the notation (7) and the directional derivative:

$$D_{-T_{\xi_0(\eta)}}\mathrm{vol}_{X_0}(\xi_0) := \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{vol}_{X_0}(\xi_0 - \epsilon T_{\xi_0(\eta)}).$$

The negative sign in front of $T_{\xi_0}(\eta)$ in the above formula is to be compatible with our later computation.

Next, we will introduce the notions of K-stability. We note that in the definition, we only look at special test configurations, in the spirit of [Tia97].

Definition 2.23 (K-stability)**.** *We say that (X, D, ξ_0) is K-semistable, if for any special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$, we have $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is nonnegative.*

We say that (X, D, ξ_0) is K-polystable, if it is K-semistable, and any special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) = 0$ is a product test configuration.

If $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is a special test configuration, we know $A(\xi_0) = A_{(X_0,D_0)}(\mathrm{wt}_{\xi_0}) > 0$. Then we see the following identity holds:

$$D_{-T_{\xi_0}\eta}\mathrm{vol}_{X_0}(\xi_0) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \widehat{\mathrm{vol}}_{X_0}(\mathrm{wt}_{\xi_0-\epsilon\eta}) \cdot \frac{1}{nA(\xi_0)^{n-1}}, \tag{8}$$

where we use the rescaling invariance of the normalized volume and $A(\xi_0) = A(\xi_0 - t \cdot T_{\xi_0}\eta)$ for $t \ll 1$ (see (7)) As a consequence, we can rewrite the Futaki invariant of a special test configuration in the following way:

$$\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) := D_{-\eta}\widehat{\mathrm{vol}}_{X_0}(\mathrm{wt}_{\xi_0}) \cdot \frac{1}{nA(\xi_0)^{n-1} \cdot \mathrm{vol}_{X_0}(\xi_0)}. \tag{9}$$

This shows that it differs from the one in [LX17, Definition 2.26] by a positive constant. It also differs from Collins-Székelyhidi's definition by a constant.

Remark 2.24. *Obviously to define the K-stability notions, we can also consider more general test configurations than the special ones. In [LX14] we proved that for the K-stability of log Fano varieties, to test on all test configurations is equivalent to only test on special test configurations.*

For log Fano cone singularities, results like [LX14] are not completely known. Nevertheless, later in this paper, we have to deal with weakly special test configurations, as they will naturally appear in our argument. Thus we need to prove a statement (see Proposition 4.3) similar to [LX14, Theorem 4], which says that for log Fano cone singularities, our definition of K-semistability is also equivalent to test on all weakly special test configurations.

Compared to the other literatures, all test configurations are considered in [CS18], whereas in [CS15, LX17] K-stability notions are only tested on special test configurations.

We will need the following simple fact, which follows from the definition of the generalized Futaki invariant applied to product test configurations:

Lemma 2.25. *Assume that the log Fano cone (X, D, ξ_0) admits a torus action by $T' \cong (\mathbb{C}^*)^{r'}$ that commutes with $\langle \xi_0 \rangle$. Let $\mathfrak{t}'$ denotes the Lie algebra of T'. Then the function*

$$\eta \mapsto \mathrm{Fut}(X_{\mathbb{C}}, D_{\mathbb{C}}, \xi_0; \eta)$$

is linear with respect to $\eta \in \mathfrak{t}'$.

Log Fano varieties

In the below, we will specialize previous definitions to the case of quasi-regular log Fano cones, which correspond to Fano projective varieties.

Definition 2.26. *Assume (S, B) is a log Fano variety. A test configuration of $(S, B, -(K_S + B))$ is a quadruple $(\mathcal{S}, \mathcal{B}, \mathcal{L}; \eta)$ with a map $\pi : (\mathcal{S}, \mathcal{B}) \to \mathbb{C}$ that satisfies the following conditions:*

(1) $\mathcal{L}$ is a π-ample $\mathbb{Q}$-line bundle and $\pi \colon \mathcal{S} \to \mathbb{C}$ is a flat family and $\mathrm{Supp}(B)$ *does not contain any component of the fiber. We denote the central fiber by (S_0, B_0, L_0).*

(2) There is a $\mathbb{C}^*$-action (with coweight η) on $(\mathcal{S}, \mathcal{B})$ such that π is $\mathbb{C}^*$-equivariant where $\mathbb{C}^*$ acts on the base $\mathbb{C}$ by multiplication and there is a $\mathbb{C}^*$-equivariant isomorphism $\phi : (\mathcal{S}, \mathcal{B}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \cong (S, B, -(K_S + B)) \times \mathbb{C}^*$, where $\mathbb{C}^*$-trivially acts on the first factor of $(S, B, -(K_S + B)) \times \mathbb{C}^*$.

Most of the time, as in the literature, we omit η in the quadruple and simply denote the test configuration by $(\mathcal{S}, \mathcal{B}, \mathcal{L})$.

Such a test configuration is called $\mathbb{Q}$-Gorenstein if $\mathcal{S}$ is G_1 and S_2,

$$K_{\mathcal{S}} + \mathcal{B} \textit{ is } \mathbb{Q}\textit{-Cartier} \quad \textit{and} \quad \mathcal{L} \sim_{\mathbb{Q}} -(K_{\mathcal{S}} + \mathcal{B}).$$

In this case, we usually just write the test configuration as $(\mathcal{S}, \mathcal{B}; \eta)$ or simply as $(\mathcal{S}, \mathcal{B})$.

A $\mathbb{Q}$-Gorenstein test configuration is called special if (S_0, B_0) is a log Fano pair with klt singularities. In this case, we say that (S_0, B_0) is a special degeneration of (S, B).

A test configuration$(\mathcal{S}, \mathcal{B}, \mathcal{L}; \eta)$ is called a product one if there is an isomorphism

$$(\mathcal{S}, \mathcal{B}, \mathcal{L}) \cong (S, B, -(K_S + B)) \times \mathbb{C} \qquad \textit{such that } \eta = \eta_0 + t\partial_t$$

where η_0 is a coweight vector on some torus group T acting on (S, B) and $t\partial_t$ is the coweight corresponding to the $\mathbb{C}^*$ factor. In this case, we will denote $(\mathcal{S}, \mathcal{B}, \mathcal{L}; \eta)$ simply by $(S_{\mathbb{C}}, B_{\mathbb{C}}; \eta)$.

For a test configuration of a log Fano variety, by trivially adding a copy over $\{\infty\}$, we can take the intersection formula (see [Wan12, Oda13]) of the generalized Futak invariant as the definition. More precisely, for any test configuration of $(\mathcal{S}, \mathcal{B})$, we can glue it with a trivial family of $(S, B) \times \mathbb{P}^1 \setminus \{0\}$ along $(S, B) \times \mathbb{C}^*$ to get $(\bar{\mathcal{S}}, \bar{B})$ over $\mathbb{P}^1$ and denote by $\bar{\mathcal{L}} \sim_{\mathbb{Q}} -(K_{\bar{\mathcal{S}}/\mathbb{P}^1} + \mathcal{B})$.

Definition 2.27 (Generalized Futaki invariants)**.** *For any $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{S}, \mathcal{B}, \mathcal{L}; \eta)$ of (S, B), we define the generalized Futaki invariant*

$$\mathrm{Fut}(\mathcal{S}, \mathcal{B}; \eta) := -\frac{\bar{\mathcal{L}}^{\cdot n}}{n(-(K_S + B))^{\cdot n-1}}.$$

By the intersection formula (see [Wan12, Oda13]), the above definition of the generalized Futaki invariants coincides with the one in [Don02].

Definition 2.28 (K-stability, see [Tia97, Don02, LX14])**.** *We say that (S, B) is K-semistable, if the generalized Futaki invariant* $\mathrm{Fut}(\mathcal{S}, \mathcal{B}; \eta)$ *is nonnegative for any special test configurations. We say that (S, B) is K-polystable, if it is K-semistable, and any special test configuration $(\mathcal{S}, \mathcal{B}, \mathcal{L}; \eta)$ with* $\mathrm{Fut}(\mathcal{S}, \mathcal{B}, \mathcal{L}; \eta) = 0$ *is a product test configuration.*

Remark 2.29. *We choose to work specifically on $\mathbb{Q}$-Gorenstein test configurations $(\mathcal{S}, B, \mathcal{L})$, since it fits into our study on log Fano cones. By [LX14], we know for a log Fano variety, working on this intermediate generality of test configurations yields the same stability notions as working either only on special test configurations or on all test configurations.*

Given a $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{S}, \mathcal{B}, \mathcal{L}; \eta)$, by choosing λ such that $\lambda(K_{\mathcal{S}}+\mathcal{B})$ is Cartier, we can get a $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ of $(X, D) := C(S, B, -\lambda(K_S+B))$ by letting $(\mathcal{X}, \mathcal{D}) = C(\mathcal{S}, \mathcal{B}; -\lambda\mathcal{L})$, $\xi_0 = u\partial_u$ the canonical rescaling vector on $\mathcal{X}$ where u is an affine coordinate on the line bundle $\lambda\mathcal{L}$, and letting η also denote its canonical lifting from $\mathcal{S}$ to $\mathcal{X}$ that corresponds to the pull back of pluri-log-canonical forms (see [Li17b, Page 3186-3187].

Lemma 2.30 (see [CS18, Theorem 4] and [Li17b, Lemma 6.20])**.** *Notations as above. If $(\mathcal{S}, \mathcal{B}; \eta)$ is a $\mathbb{Q}$-Gorenstein test configurations, then*

$$\mathrm{Fut}(\mathcal{S}, \mathcal{B}; \eta) = \mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta).$$

Proof. With the above choice of ξ_0, $A(\xi_0) = \lambda^{-1}$. Since η is the canonical lifting, we have $A(\eta) = 0$ so that $T_{\xi_0}(\eta) = \frac{A(\xi_0)}{n}\eta = \frac{\lambda^{-1}}{n}\eta$ (see (7)). So we get:

$$D_{-T_{\xi_0(\eta)}}\mathrm{vol}_{X_0}(\xi_0) = \frac{\lambda^{-1}}{n}\left.\frac{d\mathrm{vol}_{X_0}(\xi_0 - t\eta)}{dt}\right|_{t=0} = \lambda^{-1}\lim_{m\to+\infty}\frac{w_m}{mN_m} = -\frac{\lambda^{n-1}\bar{\mathcal{L}}^{\cdot n}}{n},$$

where w_m is the weight of the $\langle\eta\rangle$ action on $H^0(S_0, -m\lambda(K_{S_0} + B_0))$. The second identity follows from [CS18, Theorem 4] (see also the calculation in [Li17b, Proof of Lemma 6.20]). For the last identity, see [BHJ17, Theorem 5.3]. Dividing both sides by $\mathrm{vol}_{X_0}(\xi_0) = \lambda^{n-1}(-(K_S+B))^{\cdot n-1}$ we get the identity. $\square$

The above lemma says that the definition 2.27 is compatible with the generalized Futaki invariants for log Fano cones in Definition 2.22. Thus Definition 2.23 specializes to Definition 2.28. It is well known that if we have a product test configuration induced by a vector field coming from a $\mathbb{C}^*$-action on (S, B), then the generalized Futaki invariant defined above becomes the classical Futaki invariant. It also follows from Lemma 2.25 that

Lemma 2.31. *Assume a log Fano variety (S, B) admits a torus action by $T \cong (\mathbb{C}^*)^r$. Let $\mathfrak{t}$ be the Lie algebra of T. Then the Futaki invariant $\eta \mapsto \mathrm{Fut}(S\times\mathbb{C}, B\times\mathbb{C}; \eta)$ is a linear function with respect to $\eta \in \mathfrak{t}$.*

3 Case of log Fano pairs

In this section, we will focus on the stability of log Fano pairs. More concretely we will construct common degenerations of two K-semistable degeneration of a log Fano variety, as well as investigate the equivariant K-stability for a torus action. A point we want to note is that even in this case of log Fano varieties, we find it more flexible to work on the associated log Fano cones in order to use a combination of techniques from the minimal model program and results on normalized volumes. The study will be generalized to log Fano cones later. However, we believe that treating the case of log Fano pairs first will help the reader to more easily get the main idea.

3.1 K-semistable degeneration of K-semistable log Fano pair

We will need the following lemma which allows us to reduce a two-step equvariant degeneration to a single equivariant degeneration. The idea of its proof is similar to the one used in [LX16, Section 6]. In fact, the proof is a mimic of the argument in the classical GIT situation, but replacing Kempf's instability theorem [Kem78, Corollary 4.5] by [LX16, Theorem 1.4].

Lemma 3.1. *Let $(\mathcal{S}, \mathcal{B}, \eta)$ be a special test configuration of a K-semistable log Fano variety (S, B) with the central fiber (S_0, B_0). Suppose that $\mathrm{Fut}(\mathcal{S}, \mathcal{B}) = 0$. Then (S_0, B_0) is a K-semistable log Fano variety.*

Proof. Suppose (S_0, B_0) is not K-semistable, then by [LX16, Theorem 1.4] there is a special test configuration $(\mathcal{S}', \mathcal{B}') := (\mathcal{S}', \mathcal{B}', \eta')$ with a central fiber (S'_0, B'_0) such that

$$\mathrm{Fut}(\mathcal{S}', \mathcal{B}', \eta') < 0,$$

and $(\mathcal{S}', \mathcal{B}', \eta')$ is equivariant with respect to the $\mathbb{C}^*$-action corresponding by η.

We can assume $(\mathcal{S}, B)$ (resp. $(\mathcal{S}', B')$) are $\mathbb{C}^*$-equivariantly embedded into $\mathbb{P}^N \times \mathbb{C} \times \{0\}$ (resp. $\mathbb{P}^N \times \{0\} \times \mathbb{C}$). By abuse of notations, we denote $\eta : \mathbb{C}^* \to \mathrm{SL}(N+1)$ (resp. $\eta' : \mathbb{C}^* \to \mathrm{SL}(N+1)$) to be the 1-parameter subgroup (1-PS) generated by η (resp. η'). Then η commutes with η', or equivalently $[\eta, \eta'] = 0$. Let $\Theta = mB$ for some sufficiently divisible positive integer m such that Θ is integral.

$$\mathrm{Hilb}(S,\Theta) \in \mathbb{H}^{N,P,p} := \{(\mathrm{Hilb}(S), \mathrm{Hilb}(\Theta)) \in \mathrm{Hilb}(\mathbb{P}^N, P) \times \mathrm{Hilb}(\mathbb{P}^N, p) \mid \Theta \subset S \subset \mathbb{P}^N\} \tag{10}$$

where $P(k) = h^0(\mathcal{O}_S(k))$ and $p(k) = h^0(\mathcal{O}_\Theta(k))$ for $k \gg 1$ are the Hilbert polynomials for $(S,\Theta) \subset \mathbb{P}^N \times \mathbb{P}^N$. The $\mathrm{SL}(N+1)$-action on $\mathbb{P}^N$ induces an action on $\mathbb{H}^{N,P,p}$. We then have the following convergence:

$$\mathrm{Hilb}(S_0, \Theta_0^*) = \lim_{t\to 0} \eta(t) \cdot \mathrm{Hilb}(S,\Theta).$$

We remark $\Theta_0 := mB_0$ is not necessarily the same as the scheme Θ_0^* due to the possible appearance of embedded points on Θ_0^*. However, we have the inclusion of the ideal sheaves $I_{\Theta_0^*} \subset I_{\Theta_0}$, with the support of the cokernel being of codimension at least two on S_0. We can similarly define $p'(k) = h^0(\mathcal{O}_{\Theta_0}(k))$ and have the following convergence:

$$\begin{aligned}\mathrm{Hilb}(S_0', \Theta_0') &= \lim_{t\to 0} \eta'(t) \cdot \mathrm{Hilb}(S_0, \Theta_0) \subset \mathbb{H}^{N,P,p'}\\ \mathrm{Hilb}(S_0', \Theta_0'^*) &= \lim_{t\to 0} \eta'(t) \cdot \mathrm{Hilb}(S_0, \Theta_0^*) \subset \mathbb{H}^{N,P,p}.\end{aligned}$$

Therefore, we have the inclusion of the ideal sheaves $I_{\Theta_0'^*} \subset I_{\Theta_0'}$, and the codimension of the support of the cokernel is at least two on S_0'.

Our goal now is to construct a new test configuration $(\mathcal{S}'', \mathcal{B}'')$ of (S,B) with a special fiber (S_0', B_0') such that $\mathrm{Fut}(\mathcal{S}'', \mathcal{B}'') < 0$, contradicting to our assumption that (S,B) is K-semistable.

Notice that the action of $\mathbb{C}^* \times \mathbb{C}^* \cong \langle \eta \rangle \times \langle \eta' \rangle < \mathrm{SL}(N+1)^2$ on $\mathbb{H}^{N,P,p}$ induces a $\mathbb{C}^* \times \mathbb{C}^*$-equivariant maps f and $\hat{\phi}$ given as follows

$$\begin{tikzcd}
 & & \mathcal{G} \arrow[d, "f"] \arrow[dr, "\hat{\phi}"] & \\
\mathbb{C}^* \arrow[r, "\tau"] \arrow[urr, "\hat{\tau}"] & \mathbb{C}^* \times \mathbb{C}^* \arrow[r, hook] \arrow[rr, bend right, "(\eta,\eta')\cdot\mathrm{Hilb}(S,\Theta)"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, dashed, "\phi"] & \mathbb{H}^{N,P,p},
\end{tikzcd} \tag{11}$$

where $\mathcal{G}$ is the graph of ϕ and f is a $(\mathbb{C}^* \times \mathbb{C}^*)$-equivariant blow-up and $\phi(0,0) = \mathrm{Hilb}(S_0', \Theta_0'^*)$. Now we introduce the 1-PS

$$\begin{array}{cccc} \tau : & \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \times \mathbb{C}^* \\ & t & \longmapsto & (t^k, t) \end{array} \tag{12}$$

then for $k \gg 1$ its lift $\hat{\tau}$ satisfies

$$\hat{\phi} \circ \hat{\tau}(0) = \lim_{t\to 0} \hat{\phi} \circ \hat{\tau}(t) = \phi(0,0) = \mathrm{Hilb}(S_0', \Theta_0'^*).$$

Let $(\mathcal{S}'', \tilde{\Theta}'')$ be the flat family obtained by pulling pack the universal family $(\mathcal{S}^{\mathrm{Hilb}}, \mathcal{B}^{\mathrm{Hilb}}) \to \mathbb{H}^{N,P,p}$ via $\hat{\tau}$, and let $\mathcal{B}'' := \frac{1}{m}\tilde{\Theta}''$. Then $(\mathcal{S}'', \mathcal{B}'')$ is a special test configuration and we have

$$\begin{aligned}\mathrm{Fut}(\mathcal{S}'', \mathcal{B}'') &= \mathrm{Fut}(S_0', B_0'; k\eta + \eta')\\ &= \mathrm{Fut}(S_0', B_0'; k\eta) + \mathrm{Fut}(S_0', B_0'; \eta')\\ &= \mathrm{Fut}(S_0, B_0; k\eta) + \mathrm{Fut}(S_0', B_0'; \eta')\\ &= k \cdot \mathrm{Fut}(\mathcal{S}, \mathcal{B}) + \mathrm{Fut}(\mathcal{S}', \mathcal{B}')\\ &= 0 + \mathrm{Fut}(\mathcal{S}', \mathcal{B}') < 0,\end{aligned}$$

where we used the linearity of the Futaki invariant (cf. Lemma 2.31) in the second identity. Hence $(\mathcal{S}'', \mathcal{B}'')$ is the test configuration we are looking for and our proof is completed. □

3.2 Common degenerations of log Fano pairs

The main technical theorem of this section is the following.

Theorem 3.2. *Let (S, B) be an $(n-1)$-dimensional K-semistable log Fano variety. If there are special test configurations $(\mathcal{S}^{(i)}, \mathcal{B}^{(i)})$ $(i = 1, 2)$ of (S, B) with central fibers $(S_0^{(1)}, B_0^{(1)})$ and $(S_0^{(2)}, B_0^{(2)})$ such that $\mathrm{Fut}(\mathcal{S}^{(i)}, \mathcal{B}^{(i)}) = 0$, then there are two special test configurations $(\mathcal{S}'^{(i)}, \mathcal{B}'^{(i)})$ of $(S_0^{(i)}, B_0^{(i)})$ with isomorphic central log Fano fibers (S'_0, B'_0) such that $\mathrm{Fut}(\mathcal{S}'^{(i)}, \mathcal{B}'^{(i)}) = 0$.*

We remark that Theorem 3.2 should be regarded as an analogy of a corresponding statement in classical geometric invariant theory (GIT). As an immediate consequence we have the following:

Corollary 3.3. *In the above notion, if we assume further that $(S_0^{(1)}, B_0^{(1)})$ is K-polystable, then there is a special test configuration of $(S_0^{(2)}, B_0^{(2)})$ with generalized Futaki invariant 0 and central fiber isomorphic to $(S_0^{(1)}, B_0^{(1)})$.*

Proof of Theorem 3.2. As $(\mathcal{S}^{(i)}, \mathcal{B}^{(i)})$ $(i = 1, 2)$ are special test configurations, $(S_0^{(1)}, B_0^{(1)})$ and $(S_0^{(2)}, B_0^{(2)})$ are log Fano varieties. Consider the cone $(X, D) = C(S, B; -\lambda(K_S+B))$ over S and similarly $(X_0^{(i)}, D_0^{(i)}) = C(S_0^{(i)}, B_0^{(i)}; -\lambda(K_{S_0^{(i)}} + B_0^{(i)}))(i = 1, 2)$ for some sufficiently divisble λ. Denote the corresponding degeneration of X to $X_0^{(i)}$ over $\mathbb{C}$ to be $\mathcal{X}^{(i)}$, then we get special test configurations $(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \eta^{(i)})$ of (X, D, ξ_0), where $\mathcal{D}^{(i)}$ is the cone over $\mathcal{B}^{(i)}$ and ξ_0 is from the natural $\mathbb{C}^*$-action on the cone.

From [BHJ17, Definition 4.4], we know that the central fiber $S_0^{(1)}$ of the special degeneration $\mathcal{S}^{(1)}$ induces a valuation $w' := q \cdot \mathrm{ord}_F$ for some divisor F over S. Let ord_S denote the canonical divisorial valuation associated to the exceptional divisor, which is isomorphic to S, obtained by blowing up the vertex. Assume $\mu : \tilde{S} \to S$ is a birational morphism such that the divisor F is on $\tilde{S}$ and $(\tilde{S}, F)$ is log smooth. Let $\tilde{X} \to X$ be the resolution given by the total space of the line bundle of $\mu^*(\lambda(-K_X - B))$ over $\tilde{S}$. Then following [Li17b, Page 3181-3182], we denote by $a_1 = -\lambda(A_{(S,B)\times\mathbb{C}}(S_0^{(1)}) - 1)$ and let w_ϵ be the quasi-monomial valuation on the model $(\tilde{X}, \tilde{S} + \tilde{F})$ with weight $(1 + \epsilon a_1, \epsilon q)$ with respect to $\tilde{S}$ and the pull back $\tilde{F}$ of F by $\tilde{X} \to \tilde{S}$ (see [Li17b, Definition 6.12]). We choose ϵ^* such that $1 + \epsilon a_1 > 0$ for any $\epsilon \in [0, \epsilon^*)$. Then w_ϵ is centered at the vertex of X. By [Li17b, Proposition 6.16], we have the identity:

$$A_{(X,D)}(v_k) = kA_{(X,D)}(w_{1/k}) = k \cdot A_{(X,D)}(\mathrm{ord}_S) = k \cdot \lambda^{-1}. \tag{13}$$

For $\mathbb{N} \ni k \gg 1$, let $v_k = k \cdot w_{1/k}$. Then $v_k = d \cdot \mathrm{ord}_{E_k}$ is a multiple of a divisorial valuation ord_{E_k} for some $d \in \mathbb{Z}_{>0}$. As a valuation, we can describe v_k explicitly as follows (see [Li17b, (57)]). For any $f \in H^0(S, -m\lambda(K_S + B))$,

$$v_k(f) = km + \mathrm{ord}_{S_0^{(1)}}(\bar{f}). \tag{14}$$

where $\bar{f}$ is the meromorphic section of $m\mathcal{L} \to \mathcal{S}$ obtained by pulling back f via the map

$$(\mathcal{S} \backslash S_0^{(1)}, m\mathcal{L}) \cong (S \times \mathbb{C}^*, m \cdot p_1^* L) \xrightarrow{p_1} (S, mL).$$

Lemma 3.4. *Notations as above, for $k \gg 1$, the divisor E_k corresponding to v_k is a Kollár component with an associated model $Y_k \to (X, D)$. Moreover, the special test configuration $(\mathcal{X}^{(1)}, \mathcal{D}^{(1)})$ is given by the special test configuration associated to E_k (in the sense of Definition 2.20) up to a base change. In particular $(\mathcal{S}^{(1)}, \mathcal{B}^{(1)})$ can be recovered by the model $E_k \to Y_k \to (X, D)$.*

Proof. For simplicity, we denote $L = -\lambda(K_S + B)$. By [BHJ17, Proposition 2.15] (see also Lemma 2.18), we know that $\mathcal{X}^{(1)}$ is given by

$$\mathrm{Spec}_{\mathbb{C}[t]}\left(\bigoplus_{m\in\mathbb{N}}\left(\bigoplus_{j\in\mathbb{Z}} t^{-j}\mathcal{F}^j H^0(S, mL)\right)\right) =: \mathrm{Spec}_{\mathbb{C}[t]}(\mathcal{R}^{(1)})$$

where $\mathcal{F}^j H^0(S, mL)$ is given by:

$$\mathcal{F}^j H^0(S, mL) = \left\{ s \in H^0(S, mL) \mid t^{-j}\bar{s} \in H^0(\mathcal{S}, m\mathcal{L}) \right\}.$$

Therefore $X_0^{(1)}$ is isomorphic to

$$\mathrm{Spec}\left(\bigoplus_{j\in\mathbb{N}} \left(\bigoplus_{m\in\mathbb{N}} \mathcal{F}^j H^0(S, mL)/\mathcal{F}^{j+1} H^0(S, mL) \right) \right),$$

and the $(\mathbb{C}^*)^2$-action on $X_0^{(1)}$ is induced by the two gradings.

On the other hand, $f \in \mathcal{F}^j H^0(S, m\lambda(-K_S - B))$ if and only if $\mathrm{ord}_{S_0^{(1)}}(\bar{f}) \geq j$ which by [Li17b, (57)] is equivalent to

$$v_k(f) \quad = \quad mk + \mathrm{ord}_{S_0^{(1)}}(\bar{f}) \geq mk + j \ .$$

In other words, the valuative ideal $\mathfrak{a}_p(v_k)$ of $v_k = d \cdot \mathrm{ord}_{E_k}$ is determined by:

$$f \in H^0(S, m\lambda(-K_S - B)) \cap \mathfrak{a}_p(v_k) \text{ if and only if } f \in \mathcal{F}^{p-mk} H^0(S, m\lambda(-K_S - B)).$$

Since $v_k \in \mathrm{Val}_{X,x}$ is $\mathbb{C}^*$-invariant, we have the identity:

$$\mathrm{gr}_{v_k} R = \bigoplus_{p\in d\mathbb{Z}} \bigoplus_{m} \mathcal{F}^{p-mk} H^0(S, mL)/\mathcal{F}^{p+1-mk} H^0(S, mL).$$

Let $\xi_{\frac{1}{k}} := \xi_0 - \frac{1}{k}\eta$. For an element

$$\bar{f} \in \mathcal{F}^j H^0(S, mL)/\mathcal{F}^{j+1} H^0(S, mL),$$

its weight vector is $\alpha = (m, -j)$ and $\langle \xi_{\frac{1}{k}}, \alpha \rangle = m + \frac{j}{k}$. Thus

$$\mathrm{Proj}(\mathrm{gr}_{\mathrm{ord}_{E_k}} R) \cong \mathrm{Proj}(\mathrm{gr}_{v_k} R)$$

is the quotient of $X_0^{(1)}$ by the $\mathbb{C}^*$-action generated by $\xi_{\frac{1}{k}}$ (see Definition 2.13). So we have: $(E_k, B_k) := (X_0^{(1)}, D_0^{(1)})/\langle \xi_{\frac{1}{k}} \rangle$ (where B_k includes the orbifold locus) and E_k can be extracted over X. Since $(X_0^{(1)}, D_0^{(1)})$ has klt singularities, (E_k, B_k) is a log Fano variety which has klt singularities and hence is a Kollár component over X by the inversion of adjunction.

To see that last statement, note that we can rewrite $\mathcal{R}^{(1)}$ as:

$$\begin{aligned} \mathcal{R}^{(1)} \quad &= \quad \bigoplus_{j\in\mathbb{Z}} \bigoplus_{m\in\mathbb{N}} t^{-j} \mathfrak{a}_{mk+j}(v_k) \cap H^0(S, mL), \\ &= \quad \bigoplus_{p\in\mathbb{Z}} \bigoplus_{m\in\mathbb{N}} t^{-p+mk} \mathfrak{a}_p(v_k) \cap H^0(S, mL), \end{aligned}$$

which is isomorphic to the extended Rees algebra of $\mathfrak{a}_\bullet(v_k)$:

$$\bigoplus_{p\in\mathbb{Z}} \bigoplus_{m\in\mathbb{N}} t^{-p} \mathfrak{a}_p(v_k) \cap H^0(S, mL) = \bigoplus_{p\in\mathbb{Z}} t^{-p} \mathfrak{a}_p(v_k). \tag{15}$$

Indeed, it is easy to verify that the map $t^{-p+mk} f \mapsto t^{-p} f$ for any $f \in \mathfrak{a}_p \cap H^0(S, mL)$ is an isomorphism of the two algebras. On the other hand, the extended Rees algebra of ord_{E_k} is given by:

$$\bigoplus_{q\in\mathbb{Z}} u^{-q} \mathfrak{a}_q(\mathrm{ord}_{E_k}) = \bigoplus_{p\in d\mathbb{Z}} u^{-\frac{p}{d}} \mathfrak{a}_p(v_k).$$

From this we see that $\mathcal{X}^{(1)} = \mathcal{Y} \times_{\mathbb{C}, t\mapsto t^d} \mathbb{C}$ where $\mathcal{Y}$ is the test configuration associated to ord_{E_k} in the sense of Definition 2.20. $\square$

In the proof of Lemma 3.4, there is a rank 2 torus $(\mathbb{C}^*)^2$ acting on $X_0^{(1)}$, such that if we let ξ_0 be the coweight vector $(1,0)$, then $X_0^{(1)}/\langle\xi_0\rangle \cong S_0^{(1)}$, and the action by the coweight $(0,1)$ is induced by the action on $S_0^{(1)}$ from the test configurational $\mathcal{S}^{(1)}$ structure. We construct a ray $\xi_\epsilon = \xi_0 - \epsilon\eta$, where η corresponds the action with coweight $(0,1)$. Then any ξ_ϵ gives a quasi-monomial valuation wt_ϵ on $X_0^{(1)}$ (see (6)). Moreover, for $\epsilon \in [0,\epsilon^*)$, it also induces a sequence of quasi-monomial valuations w_ϵ in X which is contained in $\mathrm{Val}_{X,x}$ (see [LX17, Proof of Theorem 3.5]). Our proof in Lemma 3.4 just gives a verification of the divisorial valuation, which can be easily extended to the general case.

Furthermore, as proved in [Li17b, Lemma 6.20] or [LX17, Section 2.4], if we define $f(\epsilon) := \widehat{\mathrm{vol}}(w_\epsilon)$, then it is a smooth convex function on $[0,\epsilon^*)$ with $0<\epsilon^* \ll 1$ such that $f(0) = \widehat{\mathrm{vol}}(v)$ and

$$\begin{aligned} f'(0) &= \frac{d}{d\epsilon}\widehat{\mathrm{vol}}_X(w_\epsilon)\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\widehat{\mathrm{vol}}_{X_0}(\mathrm{wt}_\epsilon)\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \widehat{\mathrm{vol}}_{X_0}(\xi_0 - \epsilon\eta) \\ &= C\cdot \mathrm{Fut}(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta^{(1)}), \end{aligned}$$

where the last identity follows from (9) and the constant

$$C = n \cdot A_{X_0}(\mathrm{wt}_{\xi_0})^{n-1}\cdot \mathrm{vol}(\xi_0) > 0.$$

Lemma 3.5. *For k sufficiently large, the model $Y_k \to X$ extracting E_k can be degenerated along $\mathcal{X}^{(2)}$ to obtain a model $\mu : \mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$ over $\mathbb{C}$ with an exceptional divisor $\mathcal{E}_k$ such that the following properties hold true:*

1. *There is the following isomorphism which is equivariant with respect to the $\mathbb{C}^*$-action generated by $\eta^{(2)}$:*
$$(\mathcal{Y}_k^{(2)}, \mathcal{E}_k^{(2)}) \times_{\mathbb{C}} \mathbb{C}^* \cong (Y_k, E_k)\times \mathbb{C}^*.$$
2. *$(\mathcal{Y}_k^{(2)}, \mu_*^{-1}\mathcal{D}^{(2)} + \mathcal{E}_k^{(2)})$ locally stable over $\mathbb{C}$.*

Proof. For a fixed sufficiently large k, denote by I_k the $\mathfrak{m}$-primary ideal over $x\in X$ induced by E_k which is the push forward of $\mathcal{O}(-mE_k)$ for a fixed sufficiently divisible m. Let

$$c_k = \mathrm{lct}(I_k; X, D) =: \mathrm{lct}(I_k)$$

be its log canonical threshold. Then because E_k is a Kollár component, we have:

$$f\left(\frac{1}{k}\right) = \widehat{\mathrm{vol}}(\mathrm{ord}_{E_k}) = \mathrm{mult}(I_k)\cdot c_k^n.$$

Note that because of the rescaling invariance of the normalized multiplicities $\mathrm{mult}(I_k)\cdot \mathrm{lct}^n(I_k)$, we can replace I_k by its powers and the normalized multiplicities do not change, so we do not specifically denote m.

Since $f'(0) = C\cdot \mathrm{Fut}(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta) = 0$, we have

$$f\left(\frac{1}{k}\right) = f(0) + O\left(\frac{1}{k^2}\right).$$

Fix k, for each $l\geq 1$, as in [LX16, Lemma 4.1], we can construct a graded sequence of ideals $\mathfrak{A}_\bullet = \{\mathfrak{A}_l\}$ on $\mathcal{X}^{(2)}$ such that

$$\mathfrak{A}_l \otimes_{\mathbb{C}[t]} \mathbb{C}[t,t^{-1}] \cong I_k^l[t,t^{-1}] \quad \text{and} \quad \mathfrak{A}_l \otimes_{\mathbb{C}[t]} (\mathbb{C}[t]/(t)) \cong \mathbf{in}(I_k^l),$$

where $\{\mathbf{in}(I_k^l)\}$ is the graded sequence of ideals consisting of initial ideals of the sequence $\{I_k^l\}_l$ for the $\mathbb{C}^*$-degeneration of X to $X_0^{(2)}$. To simplify the notations, we just denote

$$\mathfrak{b}_{k,\bullet} = \{\mathfrak{b}_{k,l}\}_l = \{\mathbf{in}(I_k^l)\}.$$

Claim 3.6. *For any $\epsilon > 0$, we can find k sufficiently large and δ sufficiently small satisfying:*

$$A(E_k; X, D + c'_k I_k) < \epsilon/2 \qquad \text{and} \qquad c'_k < \operatorname{lct}(\mathfrak{b}_{k,\bullet}; X_0^{(2)}, D_0^{(2)})$$

with $c'_k := c_k(1-\delta)$.

Proof of Claim 3.6. To prove the claim, we first note that, by using $A(E_k, X, D + c_k I_k) = 0$ and identity (13):

$$A(E_k; X, D + (1-\delta)c_k I_k) = \delta \cdot A_{(X,D)}(E_k) = \delta \cdot k \cdot A_{(X,D)}(\operatorname{ord}_S).$$

On the other hand, since $(X_0^{(2)}, D_0^{(2)})$ is K-semistable by Lemma 3.1, we know that $f(0) = \operatorname{vol}(x^{(2)}, X_0^{(2)}, D_0^{(2)})$ (see Theorem 2.5), where $x^{(2)}$ is the vertex. Therefore,

$$\begin{aligned} f(0) &\leq \operatorname{lct}(\mathfrak{b}_{k,\bullet}; X_0^{(2)}, D_0^{(2)})^n \cdot \operatorname{mult}(\mathfrak{b}_{k,\bullet}) \\ &\leq c_k^n \cdot \operatorname{mult}(I_k) \\ &= f\left(\frac{1}{k}\right) = f(0) + O\left(\frac{1}{k^2}\right), \end{aligned}$$

where we have used Proposition 2.4 for the first inequality, and the non-increasing of log canonical thresholds under specialization as well as $\operatorname{mult}(\mathfrak{b}_{k,\bullet}) = \operatorname{mult}(I_k)$ for the second inequality.

We get the inequality:

$$\frac{\operatorname{lct}(\mathfrak{b}_{k,\bullet}; X_0^{(2)}, D_0^{(2)})}{c_k} \geq \left(\frac{f(0)}{f(1/k)}\right)^{1/n}.$$

Since $(1 + O(\frac{1}{k^2}))^{\frac{1}{n}}$ is also of the order $(1 + O(\frac{1}{k^2}))$, for any fixed ϵ, there exists $K_0 \gg 0$ such that for any $k \geq K_0$,

$$\left(\frac{f(0)}{f(1/k)}\right)^{1/n} \geq 1 - \frac{\epsilon}{4k \cdot A_{X,D}(\operatorname{ord}_S)}.$$

Now if we choose δ to be:

$$\delta = \frac{\epsilon}{2k \cdot A_{(X,D)}(\operatorname{ord}_S)},$$

then $c'_k = (1-\delta) \cdot c_k < \operatorname{lct}(\mathfrak{b}_{k,\bullet}; X_0^{(2)}, D_0^{(2)})$ and

$$A(E_k, X, D + (1-\delta)c_k I_k) = \epsilon/2. \tag{16}$$

$\square$

We may assume ϵ is less than 1. It follows from Claim 3.6 that

$$A(E_k \times \mathbb{C}; \mathcal{X}^{(2)}, \mathcal{D}^{(2)} + c'_k \mathfrak{A}_\bullet) < \epsilon/2 \qquad \text{and} \qquad c'_k < \operatorname{lct}(\mathfrak{A}_\bullet; \mathcal{X}^{(2)}, \mathcal{D}^{(2)} + X_0^{(2)}), \tag{17}$$

where we used the inversion of adjunction for the second inequality. We can then apply [BCHM10, Corollary 1.4.3] to precisely extract an irreducible divisor $\mathcal{E}_k^{(2)}$ to obtain a birational morphism $\mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$ whose restriction over $X \times \mathbb{C}^*$ is the divisor $E_k \times \mathbb{C}^*$ and $-\mathcal{E}_k^{(2)}$ is ample over $\mathcal{X}^{(2)}$.

Let $\mu\colon \mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$ denote the family obtained above with an irreducible divisor $\mathcal{E}_k^{(2)}$ for which we may assume $-K_{\mathcal{Y}_k^{(2)}} - \mu_*^{-1}\mathcal{D}^{(2)} - \mathcal{E}_k^{(2)}$ is ample over $\mathcal{X}^{(2)}$. Moreover, as $(\mathcal{Y}_k^{(2)}, \mu_*^{-1}\mathcal{D}^{(2)} + (1-\epsilon)\mathcal{E}_k^{(2)} + Y_0^{(2)})$ is log canonical, by ACC of log canonical thresholds ([HMX14, Theorem 1.1]), we may choose ϵ to be sufficiently small and independent of k such that $(\mathcal{Y}_k^{(2)}, \mu_*^{-1}\mathcal{D}^{(2)} + \mathcal{E}_k^{(2)} + Y_0^{(2)})$ is log canonical. $\square$

There is a $\mathbb{C}^* \times \mathbb{C}^* = \langle \xi_0 \rangle \times \langle \eta^{(2)} \rangle$-action on $X_0^{(2)}$. Note that $[\xi_0, \eta^{(2)}] = 0$. The ideals $\{\mathfrak{b}_{k,\bullet}\}$ is $(\mathbb{C}^*)^2$-equivariant. In fact, by definition it is clearly equivariant with respect to $\langle \eta^{(2)} \rangle$. It is also equivariant with respect to the first factor because E_k is $\langle \xi_0 \rangle$-invariant and $\mathcal{X}^{(2)}$ is $\langle \xi_0 \rangle$-equivariant.

$$
\begin{tikzcd}
(X_0^{(2)},D_0^{(2)}) \arrow[dr, dashed] \arrow[ddd, rightsquigarrow, "{(\mathcal{X}'^{(2)},\mathcal{D}'^{(2)})}"'] & & & (X,D) \arrow[lll, rightsquigarrow, "{(\mathcal{X}^{(2)},\mathcal{D}^{(2)})\leftarrow\mathcal{Y}_k^{(2)}\leftarrow\mathcal{E}_k^{(2)}}"'] \arrow[dl, dashed] \arrow[ddd, rightsquigarrow, "{(\mathcal{X}^{(1)},\mathcal{D}^{(1)})\leftarrow\mathcal{Y}_k\leftarrow\mathcal{E}_k=E_k\times\mathbb{A}^1}"] & Y_k \arrow[l] & E_k \arrow[l] \\
& (S_0^{(2)},B_0^{(2)}) \arrow[d, rightsquigarrow, "{(\mathcal{S}'^{(2)},\mathcal{B}'^{(2)})}"'] & (S,B) \arrow[l, rightsquigarrow, "{(\mathcal{S}^{(2)},\mathcal{B}^{(2)})}"'] \arrow[d, rightsquigarrow, "{(\mathcal{S}^{(1)},\mathcal{B}^{(1)})}"] & & & \\
& (S'_0,B'_0) & (S_0^{(1)},B_0^{(1)}) \arrow[l, rightsquigarrow, "{(\mathcal{S}'^{(1)},\mathcal{B}'^{(1)})}"] & & & \\
(X'_0,D'_0) \arrow[ur, dashed] & & & (X_0^{(1)},D_0^{(1)}) \arrow[lll, rightsquigarrow, "{(\mathcal{X}'^{(1)},\mathcal{D}'^{(1)})}"] \arrow[ul, dashed] & Y_{k,0} \arrow[l] & E_k. \arrow[l]
\end{tikzcd}
\tag{18}
$$

Now we apply the family version of the construction first introduced in [LX16, Section 2.4], to conclude that the model $\mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$ with relative anti-ample $\mathcal{E}_k^{(2)}$ over $\mathcal{X}^{(2)}$ yields a degeneration of $\mathcal{X}^{(2)}$ which gives a family $(\mathfrak{X}, \mathfrak{D})$ over $\mathbb{C}^2$, whose restriction over $(\mathbb{C}^*)^2 \subset \mathbb{C}^2$ is isomorphic to $(X, D) \times (\mathbb{C}^*)^2$. More precisely, if we assume $\mathcal{X}^{(2)} = \mathrm{Spec}_{\mathbb{C}[t]}(\mathcal{R}^{(2)})$ and define the extended Rees algebra:

$$
\mathfrak{R} = \bigoplus_{m\in\mathbb{Z}} \mathfrak{a}_m(\mathrm{ord}_{\mathcal{E}_k}) s^{-m} \subset \mathcal{R}^{(2)}[s, s^{-1}], \tag{19}
$$

where as before $\mathfrak{a}_m(\mathrm{ord}_{\mathcal{E}_k}) = \{f \in \mathcal{R}^{(2)},\ \mathrm{ord}_{\mathcal{E}_k}(f) \geq m\}$. Then $\mathfrak{X} = \mathrm{Spec}_{\mathbb{C}[t,s]}(\mathfrak{R})$ and $\mathfrak{D}$ is the divisor on $\mathfrak{X}$ induced by $\mathcal{D}^{(2)}$. Using the fact that $(\mathcal{Y}_k^{(2)}, \mu_*^{-1}\mathcal{D}^{(2)} + \mathcal{E}_k^{(2)})$ is locally stable over $\mathbb{C}$ (see Lemma 3.5.2), we know that $(\mathfrak{X}, \mathfrak{D}) \times_{\mathbb{C}^2} (\{0\} \times \mathbb{C})$ is a locally stable family (see Lemma 2.21).

Using the basic property of the Rees algebra (see e.g. [LX16, Section 4.1]), we see that

$$
(\mathfrak{X}, \mathfrak{D}) \times_{\mathbb{C}^2} (\mathbb{C} \times \{1\}) \cong (\mathcal{X}^{(2)}, \mathcal{D}^{(2)}).
$$

Moreover, we claim that:

$$
(\mathfrak{X}, \mathfrak{D}) \times_{\mathbb{C}^2} (\{1\} \times \mathbb{C}) \cong (\mathcal{X}^{(1)}, \mathcal{D}^{(1)}).
$$

This holds true if the morphism $\mathfrak{a}_m(\mathrm{ord}_{\mathcal{E}_k}) = \mu_*\mathcal{O}(-m\mathcal{E}_k) \to \mathfrak{a}_m(\mathrm{ord}_{E_k}) = (\mu|_{Y_k})_*\mathcal{O}(-mE_k)$ is surjective and the surjectivity is indeed true by using the vanishing $R^1\mu_*(-m\mathcal{E}_k) = 0$.

The restrictions of $(\mathfrak{X}, \mathfrak{D})$ over the two axes $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$ respectively give test configurations $(X_0^{(1)}, D_0^{(1)})$ and $(X_0^{(2)}, D_0^{(2)})$ with the same central fiber (X'_0, D'_0). We know these two test configurations are indeed weakly special because of the local stability of of $(\mathfrak{X}, \mathfrak{D})$.

The $\langle \xi_0 \rangle$-action on (X, D) extends naturally to $(\mathfrak{X}, \mathfrak{D})$ over $\mathbb{C}^2$. Moreover, $K_{\mathfrak{X}} + \mathfrak{D}$ is $\mathbb{Q}$-Cartier and admits a $(\mathbb{C}^*)^2$-equivariant nowhere-vanishing section $\mathfrak{s} \in |m(K_{\mathfrak{X}} + \mathfrak{D})|$. Then we can take the quotient of the action $(\mathfrak{X}, \mathfrak{D})$ by the $\langle \xi_0 \rangle$-action to get a pair $(\mathfrak{S}, \mathfrak{B})$. Its restrictions over the two axes $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$ respectively give test configurations $(S_0^{(1)}, B_0^{(1)})$ and $(S_0^{(2)}, B_0^{(2)})$ with the same central fiber (S'_0, B'_0). Because the generalized Futaki invariants are defined by the intersection numbers, we know the generalized Futaki invariant of the test configuration $(\mathfrak{S}, \mathfrak{B}) \times_{\mathbb{C}^2} (\mathbb{C} \times \{0\})$ degenerating $(S_0^{(1)}, B_0^{(1)})$ to (S'_0, B'_0) is 0 since the nearby fibers $(\mathfrak{S}, \mathfrak{B}) \times_{\mathbb{C}^2} (\mathbb{C} \times \{t\})$ $(t \neq 0)$ all have generalized Futaki invariants 0, and the same is true for the test configuration $(\mathfrak{S}, \mathfrak{B}) \times_{\mathbb{C}^2} (\{0\} \times \mathbb{C})$ degenerating $(S_0^{(2)}, B_0^{(2)})$ to (S'_0, B'_0).

Then the central fiber (S'_0, B'_0) will automatically be a log Fano variety since otherwise it follows from [LX14, Theorem 7] that we can construct a special test configuration of $(S_0^{(1)}, B_0^{(1)})$ with a strictly negative Futaki invariant, which contradicts to the K-semistability of $(S_0^{(1)}, B_0^{(1)})$ by Lemma 3.1.

Thus this completes the proof of Theorem 3.2. □

Combining the idea in the proof of Theorem 3.2 with the argument in [LX17, Proposition 4.17], we have the following fact.

Lemma 3.7. *Assume (S,B) is an $(n-1)$-dimensional K-semistable log Fano pair. Let $(\mathcal{S},\mathcal{B},\eta)$ be a special test configuration with the central fiber (S_0,B_0) such that* $\mathrm{Fut}(\mathcal{S},\mathcal{B},\eta)=0$. *If S admits a torus $T\cong(\mathbb{C}^*)^d$-action, then $\mathcal{S}$ admits a fiberwise $(\mathbb{C}^*)^d$-action over $\mathbb{C}$ which commutes with the $\mathbb{C}^*$-action generated by η and extends the $(\mathbb{C}^*)^d$ action on S. In particular, S_0 indeed admits a torus $\tilde{T}=T\times\mathbb{C}^*\cong(\mathbb{C}^*)^{d+1}$-action.*

Proof. By Lemma 3.1, we know (S_0,B_0) is K-semistable. Fix a sufficiently divisible λ. By Lemma 3.4, for $k\gg 1$, the special degeneration induces a Kollár component E_k over the cone

$$E_k\to Y_k\to (X,D)=C(S,B;-\lambda(K_S+B)).$$

The cone (X,D) is $\tilde{T}\cong T\times\mathbb{C}^*$-equivariant, where the first factor T-action is induced from the T-action on (S,B) and the second factor $\mathbb{C}^*$-action comes from the natural rescaling on the cone (X,D). Then it suffices to show that E_k is $\tilde{T}$-equivariant.

Pick an arbitrary integral coweight vector η', which generates a subgroup $\langle\eta'\rangle\cong\mathbb{C}^*\subset\tilde{T}$. Consider the valuative ideal $I_k=\mathfrak{a}_m(\mathrm{ord}_{E_k})$ for $m\gg 1$, and its equivariant degeneration $\{\mathfrak{b}_{k,l}\}_l$ of $\{I_k^l\}$ on the fiber of $X\times\mathbb{A}^1$ over 0 with respect to $\langle\eta'\rangle$. Then as before, we know there is a smooth function f on $[0,\epsilon^*)$ with $0<\epsilon^*\ll 1$ such that

$$f\left(\frac{1}{k}\right)=\mathrm{mult}(I_k)\cdot\mathrm{lct}(I_k)^n \qquad \text{and} \qquad f'(0)=0.$$

Then by exactly the same argument as in Claim 3.6, we know that for $k\gg 1$, we can pick a c_k' such that

$$A(E_k,X,D+c_k'I_k)<1 \qquad \text{and} \qquad c_k'<\mathrm{lct}(\mathfrak{b}_{k,\bullet};X_0,D_0).$$

Considering the $\langle\eta'\rangle$-action on $X\times\mathbb{C}$ induced by the diagonal action, it degenerates I_k^l to $\mathfrak{b}_{k,l}$. Applying [BCHM10, Corollary 1.4.3], we can construct a $\langle\eta'\rangle$-equivariant model $\mathcal{Y}_k\to X\times\mathbb{C}$ which extracts an exceptional divisor $\mathcal{E}_k'$ such that

$$(\mathcal{Y}',\mathcal{E}_k')\times_{\mathbb{C}}\mathbb{C}^*\cong(Y_k,E_k)\times\mathbb{C}^*.$$

But then $Y_k\times\mathbb{C}$ and $\mathcal{Y}_k$ are isomorphic in codimension one and both are the anti-ample models of the same divisorial valuation over $X\times\mathbb{C}$. This implies $\mathcal{Y}_k\cong Y_k\times\mathbb{A}^1$ and hence E_k are $\langle\eta'\rangle$-invariant. ☐

Lemma 3.8. *Assume (S,B) is a K-semistable log Fano pair and it has a special degeneration to (S_0,B_0) given by a test configuration $(\mathcal{S},\mathcal{B})$ with* $\mathrm{Fut}(\mathcal{S},\mathcal{B})=0$*; and (S_0,B_0) has a special degeneration to (S_0',B_0') given by a test configuration $(\mathcal{S}',\mathcal{B}')$ with* $\mathrm{Fut}(\mathcal{S}',\mathcal{B}')=0$. *Then there is a special degeneration of (S,B) to (S_0',B_0') given by a test configuration $(\mathcal{S}'',\mathcal{B}'')$ with* $\mathrm{Fut}(\mathcal{S}'',\mathcal{B}'')=0$.

Proof. By Lemma 3.7, $(\mathcal{S}',\mathcal{B}')$ is automatically $\mathbb{C}^*$-equivariant with respect to the action on (S_0,B_0) coming from $(\mathcal{S},\mathcal{B})$. Then this is proved in the proof of Lemma 3.1. ☐

3.3 Proof of main results for log Fano pairs

Proof of Theorem 1.3. Given a K-semistable log Fano pair $(S^{(0)},B^{(0)}):=(S,B)$. If it is not K-polystable, then by [LX14] we know it has a special degeneration to a log Fano pair $(S^{(1)},B^{(1)})$ which is not isomorphic to (S,B), with the generalized Futaki invariant being 0. Furthermore, $(S^{(1)},B^{(1)})$ is also K-semistable by Lemma 3.1. It follows from Lemma 3.8 that any special degeneration $(S^{(2)},B^{(2)})$ of $(S^{(1)},B^{(1)})$ with the generalized Futaki invariant 0 will be a special degeneration of (S,B) with the generalized Futaki invariant 0. By Lemma 3.7, if $S^{(i)}$ and $S^{(i+1)}$ are not isomorphic, then the dimension of the maximal torus effectively acting on $S^{(i+1)}$ is strictly larger than that for $S^{(i)}$. Thus this degeneration process has to terminate after $r\le\dim S$ steps. Then the end product is K-polystable and is also a special degeneration of (S,B).

The uniqueness directly follows from Theorem 3.2, as any test configuration $(\mathcal{S},\mathcal{B})$ which degenerates (S,B) to a K-polystable log Fano pair (S_0,B_0) automatically satisfies $\mathrm{Fut}(\mathcal{S},\mathcal{B})=0$. ☐

Proof of Theorem 1.4. It is known from [LX16] that to check K-semistablity, we only need to check the T-equivariant special test configurations. Then from K-semistability to K-polystability, it follows from Lemma 3.7. $\square$

Remark 3.9. *In the above discussion, we indeed prove that if there is a linear algebraic group G generated by subtori, e.g. connected reductive groups, then the polystable degeneration of a K-semistable G-equivariant $\mathbb{Q}$-Fano variety is always G-equivariant. However, our approach does not cover other cases, e.g. G is a finite group.*

4 General case of log Fano cones

In Section 4.1, we will generalize the techniques in Section 3 to the case of log Fano cones. This allows us to get weakly special test configurations with isomorphic central fibres and zero Futaki invariants, under similar assumption as in Theorem 3.2. In Section 4.2, we prove that these weakly special test configurations are indeed special. We prove this fact by generalizing the proof of [LX14, Theorem 4] to the setting of log Fano cone singularities, including the irregular case. Combining with [CS15], we complete the proof of Donaldons-Sun's conjecture (Theorem 1.1) and Theorem 1.2 on existence/uniqueness of K-polystable degenerations in Section 4.3.

4.1 Common degenerations of log Fano cones

Fix a K-semistable log Fano cone (X, D, ξ_0) with a torus action by $T \cong (\mathbb{C}^*)^r$. Then wt_{ξ_0} is a minimizer of $\widehat{\mathrm{vol}}_{X,D}$ by Theorem 2.5. Assume that $(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \eta^{(i)})$ $(i = 1, 2)$ are two special degenerations of (X, D, ξ_0) to $(X_0^{(i)}, D_0^{(i)}, \xi_0), (i = 1, 2)$ respectively. Recall that ξ_0 on $\mathcal{X}^{(i)}$ is just given by the natural extension of ξ_0 on $X \times \mathbb{C}^*$. By assumption $\eta^{(i)}$ has an integral coweight which can be written as the form $(\cdot, 1)$ with respect to the decomposition of $\tilde{T} := T \times \mathbb{C}^* \cong (\mathbb{C}^*)^{r+1}$. Note that the central fibers $(X_0^{(i)}, D_0^{(i)})$ $(i = 1, 2)$ admit $\tilde{T}$-actions generated by T and $\langle \eta^{(i)} \rangle$.

Theorem 4.1. *Let (X, D, ξ_0) be a K-semstable log Fano cone. With the notations in the above paragraph, assume $\mathrm{Fut}(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta^{(1)}) = 0$ and $\mathrm{Fut}(\mathcal{X}^{(2)}, \mathcal{D}^{(2)}, \xi_0; \eta^{(2)}) = 0$. Then there are weakly special test configurations $(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)}, \xi_0; \eta'^{(i)})$ of $\left(X_0^{(i)}, \mathcal{D}_0^{(i)}, \xi_0\right)$ $(i = 1, 2)$ with isomorphic central fibers such that $\mathrm{Fut}(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)}, \xi_0; \eta'^{(i)}) = 0$ for $i = 1, 2$.*

We follow a similar strategy as in Section 3.1.

Proof. We first claim that $(X_0^{(1)}, D_0^{(1)}, \xi_0)$ is K-semistable. If not, then there is a special test configuration $(\mathcal{X}_0''^{(1)}, \mathcal{D}_0''^{(1)}, \xi_0; \eta''^{(1)})$ with

$$\mathrm{Fut}(\mathcal{X}_0''^{(1)}, \mathcal{D}_0''^{(1)}, \xi_0; \eta''^{(1)}) < 0,$$

which degenerates $(X_0^{(1)}, D_0^{(1)}, \xi_0)$ to $(X_0''^{(1)}, D_0''^{(1)}, \xi_0)$. Then we claim there is a test configuration $(\tilde{\mathcal{X}}_0''^{(1)}, \tilde{\mathcal{D}}_0''^{(1)}, \xi_0; k\eta^{(1)} + \eta''^{(1)})$ for some $k \gg 0$ degenerating (X, D, ξ_0) to $(X_0''^{(1)}, D_0''^{(1)}, \xi_0)$ with the generalized Futaki invariant

$$\begin{aligned} & \mathrm{Fut}(\tilde{\mathcal{X}}_0''^{(1)}, \tilde{\mathcal{D}}_0''^{(1)}, \xi_0; k\eta^{(1)} + \eta''^{(1)}) \\ = \; & k \cdot \mathrm{Fut}(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta^{(1)}) + \mathrm{Fut}(\mathcal{X}_0''^{(1)}, \mathcal{D}_0''^{(1)}, \xi_0; \eta''^{(1)}) \\ < \; & 0, \end{aligned}$$

which is contradictory to our assumption (X, D, ξ_0) is K-semistable. Here we used the linearity of the generalized Futaki invariant from Lemma 2.25 as in the log Fano variety case. To see the existence of such test a configuration we fix a rational vector $\xi_0' \in \mathfrak{t}_{\mathbb{R}}^+$, and take the quotient, we get

$$(\mathcal{S}_0^{(1)}, \mathcal{B}_0^{(1)}) := (\tilde{\mathcal{X}}_0^{(1)}, \tilde{\mathcal{D}}_0^{(1)})/\langle \xi_0' \rangle \;\; \text{and} \;\; (\mathcal{S}_0''^{(1)}, \mathcal{B}_0''^{(1)}) := (\tilde{\mathcal{X}}_0''^{(1)}, \tilde{\mathcal{D}}_0''^{(1)})/\langle \xi_0' \rangle \,.$$

Since $[\eta^{(1)}, \eta''^{(1)}] = 0$, the proof of Lemma 3.1 shows that there is a test configuration $(\tilde{\mathcal{S}}_0''^{(1)}, \overline{B}_0''^{(1)})$ that degenerates $(X,D)/\langle\xi_0'\rangle$ to $(X_0''^{(1)}, D_0''^{(1)})/\langle\xi_0'\rangle$. Then we can take the cone back to get $(\tilde{\mathcal{X}}_0''^{(1)}, \tilde{\mathcal{D}}_0''^{(1)}, \xi_0)$. (Also see [LX17, Section 4.2] for a direct construction.)

Applying the diophantine approximation (cf. [LX17, Lemma 2.7]) of the coordinates of ξ_0, we can choose a sequence of integral vectors $\{\tilde{\xi}_k\}$ such that $|\tilde{\xi}_k - k\xi_0| \le A$ for any constant $A > 0$ where k is an infinite sequence of increasing positive integers. Consider the Kollár component E_k determined by $\tilde{\xi}_k - \eta^{(1)}$ over $x^{(1)} \in (X_0^{(1)}, D_0^{(1)}, \xi_0)$ (it is a Kollár component by Lemma 3.4). Let $I_k = \mathfrak{a}_m(\mathrm{ord}_{E_k})$ for a sufficiently divisible m depending on k. Let $c_k = \mathrm{lct}(I_k; V, B)$ and consider:

$$f(\frac{1}{k}) = \widehat{\mathrm{vol}}(\mathrm{ord}_{E_k}) = \mathrm{mult}(I_k)\cdot c_k^n.$$

Let $\tilde{T} = \langle \xi_0, \eta^{(1)}\rangle \cong (\mathbb{C}^*)^{r+1}$ be the torus generated by ξ_0 and $\eta^{(1)}$, and $\tilde{N} = \mathrm{Hom}(\mathbb{C}^*, \tilde{T})$ be the coweight lattice of $\tilde{T}$.

Since $(X_0^{(1)}, D_0^{(1)}, \xi_0)$ is K-semistable,

$$\widehat{\mathrm{vol}}(\xi) := \widehat{\mathrm{vol}}_{(X_0^{(1)}, D_0^{(1)})}(\mathrm{wt}_\xi)$$

is a smooth function of $\xi \in \tilde{\mathfrak{t}}^+_{\mathbb{R}}$ and obtains the minimum at ξ_0 (see Theorem 2.5). By (9), this also implies that for any rational vector $\eta_1 \in \tilde{N}_{\mathbb{R}}$,

$$\frac{d\, \widehat{\mathrm{vol}}(\mathrm{wt}_{\xi_0 - t\eta_1})}{dt} = C \cdot \mathrm{Fut}(X_0^{(1)} \times \mathbb{C}, D_0^{(1)} \times \mathbb{C}, \xi_0; \eta_1) = 0 \tag{20}$$

By Taylor's Remainder Theorem there is a neighborhood U of $\xi_0 \in \tilde{N}_{\mathbb{R}}$ and a positive constant $C > 0$ (independent of ξ) such that, for any $\xi \in U$, we have the inequality:

$$\widehat{\mathrm{vol}}(\xi_0) \le \widehat{\mathrm{vol}}(\xi) \le \widehat{\mathrm{vol}}(\xi_0) + C|\xi - \xi_0|^2.$$

Note that $f(\frac{1}{k}) = \widehat{\mathrm{vol}}(\frac{1}{k}\tilde{\xi}_k - \frac{1}{k}\eta^{(1)})$ by the rescaling invariance of the normalized volume. Because $\left|\frac{1}{k}\tilde{\xi}_k - \frac{1}{k}\eta^{(1)} - \xi_0\right| \le C' k^{-1}$ for $C' > 0$ independent of k, there exists $K_0 \gg 1$ such that for any $k \ge K_0$, $f(\frac{1}{k}) = f(0) + O(\frac{1}{k^2})$.

Then the same argument as in the case of the log Fano varieties using [BCHM10, Corollary 1.4.3], shows that we can find $\mu^{(2)}\colon \mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$ a morphism over $\mathbb{C}$ with a divisor $\mathcal{E}_k^{(2)}$ such that $-\mathcal{E}_k^{(2)}$ is ample over $\mathcal{X}^{(2)}$ and $(\mathcal{Y}_k^{(2)}, \mathcal{E}_k^{(2)}) \times_{\mathbb{C}} \mathbb{C}^* = (Y_k, E_k) \times \mathbb{C}^*$ where the isomorphism is equivariant with respect to the $\mathbb{C}^*$-action generated by $\eta^{(2)}$. Moreover, fixed any arbitrarily small ϵ, we can choose k sufficiently large such that the log discrepancy of E_k with respect to $(X, D + (1-\delta)\cdot I_k)$ is less than ϵ (see (16)). Then it follows from [HMX14] that $(\mathcal{Y}_k^{(2)}, \mathcal{E}_k^{(2)} + (\mu^{(2)})_*^{-1}\mathcal{D}^2)$ is locally stable over $\mathbb{C}$.

The relative extended Rees algebra gives a family $(\mathfrak{X}, \mathfrak{D})$ over $\mathbb{C}^2$, such that over $\mathbb{C} \times \{t\}$ (resp. $\{t\} \times \mathbb{C}$) ($t \neq 0$), it gives a family which is isomorphic to $(\mathcal{X}^{(1)}, \mathcal{D}^{(1)})$ (resp. $(\mathcal{X}^{(2)}, \mathcal{D}^{(2)})$). The family $(\mathfrak{X}, \mathfrak{D})$ admits a $(\mathbb{C}^*)^{r+2}$-action.

By Lemma 2.21, we get *weakly special* test configurations

$$(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)}, \xi_0; \eta'^{(i)}) \text{ of } (X_0^{(i)}, D_0^{(i)}, \xi_0) \quad (i = 1, 2)$$

with an isomorphic central fiber (X_0', D_0', ξ_0).

We claim that the generalize Futaki invariants $\mathrm{Fut}(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)}, \xi_0; \eta'^{(i)})$ are 0. Indeed, by the construction,

$$(\mathfrak{X}, \mathfrak{D}, \xi_0; \eta^{(1)})\Big|_{\mathbb{C}\times\{t\}} \cong (\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta^{(1)}).$$

It follows from our assumption that

$$\mathrm{Fut}(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta^{(1)}) = \mathrm{Fut}(X_0^{(1)}, D_0^{(1)}, \xi_0; \eta^{(1)}) = 0.$$

By the flatness of the weighted piece and $(\mathbb{C}^*)^2$ equivariance, we get for any t,

$$\mathrm{vol}_{X_0^{(1)}}(\xi_0 - t\eta^{(1)}) = \mathrm{vol}_{X_0'}(\xi_0 - t\eta^{(1)}),$$

which implies that $\mathrm{Fut}(X_0', D_0', \xi_0; \eta^{(1)}) = 0$ (see (20)). Similarly, we have $\mathrm{Fut}(X_0', D_0', \xi_0; \eta^{(2)}) = 0$.

□

By the above result, we obtain two *weakly special* test configurations $(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)}, \xi_0; \eta'^{(i)})$ with isomorphic central fibres $(X_0'^{(1)}, D_0'^{(1)}, \xi_0) \cong (X_0'^{(2)}, D_0'^{(2)}, \xi_0)$ and zero generalized Futaki invariants. In the next subsection, we are going to show that $(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)}, \xi_0; \eta'^{(i)})$ are indeed special test configurations.

4.2 Vanishing Futaki invariants and special degenerations

We will prove Proposition 4.3, which says to test K-(semi, poly)stability of a log Fano cone, although in our definition we only require to test on all special degenerations, it is indeed the same to test on all weakly special test configurations. A tool we will use is to write the generalized Futaki invariant of a weakly special configuration as the derivative of the leading coefficient of the index character (see [MSY08, CS18, CS15]).

If there are two T-equivariant weakly special test configurations

$$(\mathcal{X}^{(i)} = \mathrm{Spec}(\mathcal{R}^{(i)}), \mathcal{D}^{(i)}, \xi_0; \eta) \quad \text{of a K-semistable log Fano cone } (X, D, \xi_0),$$

with $\mathrm{Fut}(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \eta) = 0$, by Lemma 2.18, we know $\mathcal{X}^{(i)}$ is associated to a graded sequence of ideals $\mathfrak{a}_\bullet^{(i)}$ which we can assume to be primary (see Remark 2.19) as

$$\begin{aligned} & \mathrm{Fut}(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; m\xi_0' + \eta) \\ = \; & \mathrm{Fut}(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; m\xi_0') + \mathrm{Fut}(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \eta) \\ = \; & 0, \end{aligned}$$

where $\mathrm{Fut}(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \xi_0') = 0$ follows from the K-semistability of (X, D). Moreover, since the test configuration is weakly special, by Lemma 2.21 there is indeed a birational morphism $\mu^i \colon Y \to X$ with a reduced exceptional divisor E^i such that $(Y^i, E^i + (\mu^i)^{-1}_* D)$ is log canonical and $\mathfrak{a}_k^{(i)} = \mu^i_*(-kE^i)$. Therefore, we can take a normalized graph $\mu^g \colon Y^g \to X$ of $Y^1 \dashrightarrow Y^2$ over X with $p_i \colon Y^g \to Y^i$. Then for any pair (a, b) such that $\big(-ap_1^*(E^1) - bp_2^*(E^2)\big)$ is integral, by Definition 2.20, we can consider the test configuration $\mathcal{Y}_{a,b}$ of (X, D, ξ_0) induced by $\big(-ap_1^*(E^1) - bp_2^*(E^2)\big)$.

We apply the T-equivariant index character (see [CS18, Section 4] for more details) for any $\xi \in \tilde{\mathfrak{t}}^+_{\mathbb{R}} \subset \tilde{N}_{\mathbb{R}} \cong \mathbb{R}^{r+1}$ where $\tilde{\mathfrak{t}}^+_{\mathbb{R}}$ is the Reeb cone of the $\tilde{T} = T \times \mathbb{C}^*$-action and $t \in \mathbb{C}$ with the real part $\Re(t) > 0$, and define:

$$F(a, b; \xi, t) = \sum_{\alpha \in \tilde{\mathfrak{t}}^+_{\mathbb{R}}} e^{-t\alpha(\xi)} \dim R^{a,b}_\alpha(v), \tag{21}$$

where $R^{a,b}$ is the ring of the special fiber of $\mathcal{Y}_{a,b}$.

Now if we fix a prime integral vector $\xi \in t^+_{\mathbb{R}} \cap N$ such that

$$(\mathcal{X}^{(i)}, \mathcal{D}^{(i)})/\langle \xi \rangle = (\mathcal{S}^{(i)}, B^{(i)}, \mathcal{L}^{(i)}) \quad (i = 1, 2)$$

give test configurations of $(X, D)/\langle \xi \rangle = (S, B)$ with polarizations $\mathcal{L}^i$. Then the quotient of $\mathcal{Y}_{a,b}$ by $\langle \xi \rangle$ is given by the normalized graph $\mathcal{S}_{a,b}$ of $\mathcal{S}^{(1)} \dashrightarrow \mathcal{S}^{(2)}$ with morphisms $\phi_i \colon \mathcal{S}_{a,b} \to \mathcal{S}^{(i)}$ and the polarization is given by $a\phi_1^*\mathcal{L}^{(1)} + b\phi_2^*\mathcal{L}^{(2)}$.

The following statement essentially follows from [CS18, Theorem 4.10].

Proposition 4.2. *For a fixed $\xi \in \tilde{\mathfrak{t}}$ the index character $F(a, b; \xi, t)$ has a meromorphic extension to $\mathbb{C}$ with poles along the imaginary axis. Near $t = 0$ it has a Laurent series expansion:*

$$F(a, b; \xi, t) = \frac{a_0(a, b; \xi) n!}{t^{n+1}} + \frac{a_1(a, b; \xi)(n-1)!}{t^n} + \cdots, \tag{22}$$

where $a_0(a, b; \xi)$ is a polynomial of (a, b) whose coefficients depends smoothly on $\xi \in \tilde{\mathfrak{t}}^+_{\mathbb{R}}$.

Proof. It follows from [CS18, Proposition 4.3] that when ξ is rational, then a_0 coincides with the leading term of the total weight on the test configuration $\mathcal{S}_{a,b}$ constructed from the quotient log Fano pair. Since it can be represented by an intersection formula, in particular, it is a polynomial of a and b by [Wan12, Oda13].

Denote by $s = r+1$. By the proof of [CS18, Theorem 4.10], we know

$$F(a,b;\xi,t) = \frac{e^{-t(\xi_1\alpha_1+\cdots\xi_s\alpha_s)}\cdot HN_{a,b}(e^{-t\xi_1},...e^{-t\xi_s})}{\Pi_{j=1}^N(1-e^{-t(\xi_1 w_{1j}+\cdots+\xi_s w_{sj})})},$$

where $\xi=(\xi_1,...,\xi_s)\in\tilde{\mathfrak{t}}^+_{\mathbb{R}}$, $(\alpha_1,...,\alpha_s)\in\mathbb{Z}^s$ and w_{ij} $(1\le i\le s, 1\le j\le N)$ are real numbers. The leading term of the Laurent expansion is the same as the leading term of

$$\frac{HN_{a,b}(1,...,1)}{\Pi_{j=1}^N(1-e^{-t(\xi_1 w_{1j}+\cdots+\xi_s w_{sj})})}\ .$$

Since a,b only appear in the part $HN_{ab}(1,...,1)$ which does not depend on ξ, and from the case that ξ is rational, we know that $HN_{a,b}(1,...,1)$ is a polynomial of (a,b), which implies a_0 is a polynomial of (a,b). $\square$

With all these preparations, we can prove Proposition 4.3 which is a generalization of [LX14, Theorem 4] from the quasi-regular case to the general case of an arbitrary log Fano cone singularity. Although we expect the full results of special degeneration in [LX14] can be extended, here we only need the last step of the argument.

Proposition 4.3. *Let $(\mathcal{X},\mathcal{D},\xi_0;\eta)$ be a weakly special test configuration of a log Fano cone singularity (X,D,ξ_0). Then we can find a special test configuration $(\mathcal{X}',\mathcal{D}',\xi_0;\eta')$ and a positive integer m such that*

$$\mathrm{Fut}(\mathcal{X}',\mathcal{D}',\xi_0;\eta')\le m\cdot\mathrm{Fut}(\mathcal{X},\mathcal{D},\xi_0;\eta),$$

and the strict inequality holds if $(\mathcal{X},\mathcal{D},\xi_0;\eta)$ is not a special test configuration.

Proof. By Lemma 2.21, the weakly special test configuration is induced by a T-equivariant morphism $\mu\colon Y\to X$, such that the reduced exceptional divisor E is anti-ample over X and $(Y,E+\mu_*^{-1}D)$ is log canonical. Suppose $(\mathcal{X},\mathcal{D},\xi_0;\eta)$ is not special, then $(Y,E+\mu_*^{-1}D)$ is not plt. Therefore, by [LX16, Proposition 2.10], we can find a T-equivariant Kollár component S over $x\in(X,D)$ such that its log discrepancy with respect to $(Y,E+\mu_*^{-1}D)$ is 0. Denote by $\mu'\colon Y'\to X$ the plt blow extracting precisely S. So by Lemma 2.21 again, it gives a special test configuration $(\mathcal{X}',\mathcal{D}',\xi_0;\eta')$ and the base change factor m (which we omit from now on) corresponds to a multiple such that the coefficient of S in the pull back of mE is integral.

Let Y^g be the normalized graph $Y\dashrightarrow Y'$ and $p\colon Y^g\to Y$, $p'\colon Y^g\to Y'$ the natural morphisms. Then for any pair of positive integers (a,b), the divisor $bp^*E+ap'^*S$ are anti-ample, and therefore induces a test configuration $\mathcal{X}_{a,b}$ by Lemma 2.21. We take $a_0(a,b,\xi)$ as in Proposition 4.2.

Now we claim that

$$D_{-T_{\xi_0}\eta}a_0(1,0,\xi_0)=\mathrm{Fut}(\mathcal{X},\mathcal{D},\xi_0;\eta)>\mathrm{Fut}(\mathcal{X}',\mathcal{D}',\xi_0;\eta')=D_{-T_{\xi_0}\eta}a_0(0,1,\xi_0).$$

To see this we write:

$$p^*(K_Y+E+\mu_*^{-1}D)=p'^*(K_{Y'}+S+(\mu'_*)^{-1}D)+G,$$

and since the log discrepancy $A_{Y,E+\mu_*^{-1}D}(S)=0$, the negativity lemma implies that $G\ge 0$.

For any irreducible component E_i in $\mathrm{Supp}(G)$, denote by c_i its coefficient in G. In particular, from our assumption that $\mathcal{X}$ is not a special test configuration, for some component E_0 contained in $\mathrm{Supp}(E)$, its coefficient c_0 is positive. Let F_i be divisor on X_0 given by the orbifold cone $C(E_i,-E|_{E_i})$.

We take the previous construction for the two test configurations $\mathcal{X}$ and $\mathcal{X}'$. By Proposition 4.2, for a fixed ξ_0, if we define

$$f(t;\xi_0)=D_{-T_{\xi_0}\eta}a_0(1-t,t;\xi_0),$$

then the difference of the generalized Futaki invariant is of the form

$$\mathrm{Fut}(\mathcal{X}',\mathcal{D}',\xi_0;\eta')-\mathrm{Fut}(\mathcal{X},\mathcal{D},\xi_0;\eta)=\int_0^1\frac{d}{dt}f(t;\xi_0)\,dt.$$

The integrand is smooth in $[0,1]$, and the proof of [LX14, Proposition 5] shows that it is non-positive when ξ_0 is rational. Thus it is non-positive. We claim its value at 0 is

$$\left.\frac{d}{dt}f(t;\xi_0)\right|_{t=0} = -\frac{1}{2\cdot \mathrm{vol}_X(\mathrm{wt}_{\xi_0})}\sum_i c_i \cdot \mathrm{vol}_{F_i}(\mathrm{wt}_{\xi_0}) < 0. \tag{23}$$

In fact to see (23), when ξ_0 is rational, we can compute on the quotient log Fano pair, and this is given in [LX14, Page 217]. Since both sides are smooth functions on ξ_0, we know that they must be equal to each other. □

An immediate consequence is the following.

Corollary 4.4. *For a K-semistable log Fano cone singularity (X, D, ξ), if it has a weakly special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with the generalized Futaki invariant being 0, then it is a special test configuration, i.e., the central fiber is klt.*

4.3 Completion of proof of main theorems for log Fano cones

Proof of Theorem 1.2. The proof follows the same structure as the proof of Theorem 1.3.

We first prove the existence of K-polystable degenerations. In the proof of Theorem 4.1 we have shown that for any special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ of $(X^{(0)}, D^{(0)}, \xi_0) := (X, D, \xi_0)$ with $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) = 0$, the special fiber $(X^{(1)}, D^{(1)}, \xi_0)$ is K-semistable. Furthermore, any special degeneration of $(X^{(1)}, D^{(1)}, \xi_0)$ can be indeed written as a special degeneration of $(X^{(0)}, D^{(0)}, \xi_0)$. Similar to the proof of Theorem 1.3, if the K-semistable degeneration $(X^{(1)}, D^{(1)}, \xi_0)$ is not isomorphic to $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$, then $(X^{(1)}, D^{(1)}, \xi_0)$ admits an effective action by a torus $\tilde{T}$ one dimensional larger than $\dim(T)$. Therefore such step has to terminate, namely when we have the central fiber being K-polystable.

By Corollary 4.4, we can replace the words "weakly special" by "special" in the statement of Theorem 4.1. Recall that by definition special degenerations of K-polystable log Fano cone with zero generalized Futaki invariants must be product. So the uniqueness of K-polystable degeneration follows. □

Proof of Theorem 1.1. It is shown in [DS17] that there is a special test configuration $(\mathcal{W}, \xi_0; \eta)$ of of the intermediate cone (W, ξ_0) with the central fiber (C, ξ_0). Because C admits a Ricci-flat Kähler cone metric, we know C is K-polystable (see [CS15, Theorem 1.1]). In particular, $\mathrm{Fut}(\mathcal{W}, \xi_0; \eta) = 0$. Moreover by [LX17, Theorem 1.4], we know that W is K-semistable and is uniquely determined by the algebraic germ (M_∞, o).

Assume W specially degenerates to another K-polystable Fano cone C' by a special test configuration $(\mathcal{W}', \xi_0; \eta')$ with $\mathrm{Fut}(\mathcal{W}', \xi_0; \eta') = 0$. Then Theorem 4.1 implies that C and C' degenerates to a Fano cone C'' by special test configurations with generalized Futaki invariants 0. This implies $C \cong C'' \cong C'$ by the polystability of C and C'.

□

A Ding-polystability of Ricci-flat Kähler cones

In the proof of Theorem 1.1 above, we rely on the result proved in [CS15] which says that that for a log Fano cone singularity with a Ricci-flat Kähler cone metric, the generalized Futaki invariant $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) > 0$ for any non-product *special* test configuration. However, as we have seen, in our argument (see e.g. the proof of Theorem 4.1), more general test configuration will show up. Therefore in this appendix, we want to discuss the proof of a more general statement, namely for any non-product $\mathbb{Q}$-Gorenstein test configuration, the corresponding Ding invariant is positive (see Theorem A.2). This can be used to slightly modify the proof of Theorem 1.1 (see Remark A.4). We point out that our proof of Theorem A.2 follows the general strategy in [Ber15] and is slightly different from [CS15].

Definition A.1 (Ding-stability)**.** *We say that (X, D, ξ_0) is Ding-semistable, if for any $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ of (X, D, ξ_0) with the central fiber (X_0, D_0, ξ_0), its Berman-Ding invariant, denoted by $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is nonnegative, where*

$$D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) := \frac{D_{-T_{\xi_0}(\eta)}\mathrm{vol}_{X_0}(\xi_0)}{\mathrm{vol}(\xi_0)} - (1 - \mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0)).$$

We say that (X, D, ξ_0) is Ding-polystable, if it is Ding-semistable, and any $\mathbb{Q}$-Gorenstein test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) = 0$ is a product test configuration.

We immediately see that $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) = \mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ if and only if the test configuration is weakly special, and Ding-semistability (resp. Ding-polystability) implies K-semistability (resp. K-polystability). It is proved that in the log Fano pair case, they are equivalent [BHJ17, Fuj16]. Following [Ber15], it will become clear that the notions of Ding-stability fit better into our calculation.

Theorem A.2. *Assume (X, ξ_0) admits a Ricci-flat Käler cone metric. Then (X, ξ_0) is Ding-polystable among $\mathbb{Q}$-Gorenstein test configurations.*

Corollary A.3. *Assume (X, ξ_0) admits a Ricci-flat Kähler cone metric. Then (X, ξ_0) is K-polystable among all weakly special test configurations.*

Remark A.4. *Corollary A.3 could yield an alternative argument in one step of the proof of Theorem 1.1. More precisely, with notations in the proof of Theorem 1.1, let C and C' be two K-polystable degenerations of W. Then the degenerations of C and C' to C'' obtained via Theorem 4.1 are weakly special with zero Futaki invariant. We can skip Proposition 4.3 but replace [CS15, Theorem 1.1] by the stronger statement Corollary A.3, which directly implies there is no non-product weakly special test configurations of C and C' with zero Futaki invariant. Then we conclude immediately that $C \cong C'' \cong C'$.*

Let (X, ξ_0) be a Fano cone singularity with the vertex point o. Recall that this implies that X is a normal affine variety with at worst klt singularities. Moreover there is a good T action where $T \cong (\mathbb{C}^*)^r$ and $\xi_0 \in \mathfrak{t}_{\mathbb{R}}^+$. On X there exists a T-equivariant nowhere-vanishing holomorphic m-pluricanonical form $s \in |-mK_X|$. Such holomorphic form can be solved uniquely up to a constant as in [MSY08, 2.7]. In the following, we will use the following volume form on X associated to s:

$$dV_X = \left(\sqrt{-1}^{mn^2} s \wedge \bar{s}\right)^{1/m}. \tag{24}$$

Assume that (X, ξ_0) is equivariantly embedded into $(\mathbb{C}^N, \xi_0)$ with $\xi_0 = \sum_i a_i z_i \frac{\partial}{\partial z_i}$ with $a_i \in \mathbb{R}_{>0}$. Fix a reference smooth Kähler cone metric on $\mathbb{C}^N$ whose associated Reeb vector field $r\partial_r - iJ(r\partial_r) = 2\xi_0$. By its rescaling property such a radius function is C^0-comparable to $\sum_{i=1}^N |z_i|^{2/a_i}$. The restriction $\omega_X := \omega_{\mathbb{C}^N}|_X$ is a Kähler cone metric on X. Moreover $2\mathrm{Im}(\xi_0) = J(r\partial_r)$ is the Reeb vector field of $\omega_{\mathbb{C}^N}$ and ω_X. Since T acts on X, T also acts on the set of functions on X by $\tau \circ f(x) = f(\tau^{-1}x)$ for any $\tau \in T$ and $x \in X$. For convenience, we denote $X^\circ = X \setminus \{o\}$ where o is the vertex of X and define:

Definition A.5. *Denote by $PSH(X, \xi_0)$ the set of bounded real functions φ on X° that satisfies:*

(1) $\tau \circ \varphi = \varphi$ for any $\tau \in \langle \xi_0 \rangle$;

(2) $r_\varphi^2 := r^2 e^\varphi$ is a proper plurisubharmonic function on X.

We can think of functions in $PSH(X, \xi_0)$ as transversal Kähler potentials as in [DS17]. More precisely, because ∂_r generates a $\mathbb{R}_+$-action ($\mathbb{R}_+ = \{a \in \mathbb{R}; a > 0\}$) on X° without fixed points, if the link of X is defined as $Y := \{r = 1\} \cap X$, then $Y = X^\circ/\mathbb{R}_+$ and $X^\circ \cong Y \times \mathbb{R}_+$. We denote:

$$\chi = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial) \log r^2 = -\frac{1}{2} J d \log r^2, \tag{25}$$

and define:

Definition A.6. *Denote by $PSH(Y, \xi_0)$ the set of bounded real function φ on Y that satisfies:*

(1) $\tau \circ \varphi = 0$ for $\tau \in \exp(\mathbb{R} \cdot \mathrm{Im}(\xi_0))$.

(2) φ is upper semicontinuous on Y and $(d\chi + \sqrt{-1}\partial\bar{\partial}\varphi)\big|_Y \geq 0$, where the positivity is in the sense of currents.

Here we identify the function on Y with its pull back to $X^\circ \cong Y \times \mathbb{R}_+$ via the projection to the first factor. There is an isomorphism $PSH(X, \xi_0) \cong PSH(Y, \xi_0)$ by sending $\varphi \mapsto \varphi|_Y$. We will use these two equivalent descriptions in the following discussion.

Definition A.7. *We say that $r_\varphi^2 := r^2 e^\varphi$ where $\varphi \in PSH(X,\xi_0)$ is a radius function of a Ricci-flat Kähler cone metric on (X,ξ_0) if φ is smooth on X^{reg} and there exists a constant $C>0$ such that*

$$(\sqrt{-1}\partial\bar\partial r_\varphi^2)^n = C\cdot dV_X. \tag{26}$$

If we take $\mathcal{L}_{r\partial_r}$ on both sides, we get: $\mathcal{L}_{r\partial_r} dV_X = 2n dV_X$, which is also equivalent to $\mathcal{L}_{\xi_0} s = mns$. If we write

$$dV_X = 2r^{2n-1}dr\wedge\Omega_Y, \quad \text{or equivalently} \quad \Omega_Y := 2^{-1}r^{1-2n} i_{\partial_r} dV_X, \tag{27}$$

then $\mathcal{L}_{\partial_r}\Omega_Y = 0$. On the other hand, a direct computation shows that:

$$\sqrt{-1}\partial\bar\partial r_\varphi^2 = r_\varphi^2(d\chi + \sqrt{-1}\partial\bar\partial\varphi) + dr_\varphi^2 \wedge \left(\chi - \frac{1}{2}Jd\varphi\right), \tag{28}$$

Then it is easy to verify that the equation (26) is equivalent to:

$$(d\chi + \sqrt{-1}\partial\bar\partial\varphi)^{n-1}\wedge\chi = \frac{C}{n}\cdot e^{-n\varphi}\Omega_Y. \tag{29}$$

The equation (26) is the Euler-Lagrange equation for the following Ding-type functional:

Definition A.8 (see [CS15, LX17])**.** *For any function $\varphi\in PSH(X,\xi_0)$, define:*

$$D(\varphi) = E(\varphi) - \log\left(\int_X e^{-r_\varphi^2} dV\right) =: E(\varphi) + G(\varphi) \tag{30}$$

where $E(\varphi)$ is defined by its variations:

$$\delta E(\varphi)\cdot\delta\varphi = -\frac{1}{(n-1)!(2\pi)^n \mathrm{vol}_X(\xi_0)}\int_X (\delta\varphi) e^{-r_\varphi^2}(\sqrt{-1}\partial\bar\partial r_\varphi^2)^n.$$

Using the identity (28), one can verify that:

$$\delta E(\varphi)\cdot\delta\varphi = -\frac{n}{(2\pi)^n\mathrm{vol}(\xi_0)}\int_Y (\delta\varphi)(d\chi+\sqrt{-1}\partial\bar\partial\varphi)^{n-1}\wedge\chi. \tag{31}$$

As in the standard Kähler case, a consequence of this description is the following explicit expression of $E(\varphi)$ (see [DS17]):

$$E(\varphi) = -\frac{1}{(2\pi)^n\mathrm{vol}(\xi_0)}\sum_{i=0}^{n-1}\int_Y \varphi(d\chi+\sqrt{-1}\partial\bar\partial\varphi)^i\wedge(d\chi)^{n-1-i}\wedge\chi. \tag{32}$$

In the similar vein, using (27) we have the identity:

$$G(\varphi) = -\log\left(\int_Y e^{-n\varphi}\Omega_Y\right) - \log(n-1)!. \tag{33}$$

We will study the asymptotic of $E(\varphi_t)$. In the following we will denote $\mathbb{D} := \{z\in\mathbb{Z}; |z|\le 1\}$, $\mathbb{D}^* = \mathbb{D}\setminus\{0\}$ and $S^1 = \{z\in\mathbb{D}; |z|=1\}$. We will always identify the functions on X with functions on $X\times\mathbb{D}$ or $X\times\mathbb{D}^*$ by pulling back via the projection to the first factor.

Proposition A.9 (see [LX17, Lemma 5.10])**.** *Let $\varphi(x,t)=\varphi(x,|t|): X\times\mathbb{D}^*\to\mathbb{R}$ be a upper semicontinuous function such that $\varphi_t := \varphi(\cdot,|t|)\in PSH(X,\xi_0)$ for each $t\in\mathbb{D}^*$. Assume $\sqrt{-1}\partial\bar\partial(r^2e^\varphi)\ge 0$ over $X\times\mathbb{D}^*$ in the sense of currents. Then the following identity holds:*

$$\begin{aligned}\sqrt{-1}\frac{\partial^2}{\partial t\partial\bar t}E(\varphi_t)dt\wedge d\bar t &= -\frac{1}{(n+1)!(2\pi)^n\mathrm{vol}(\xi_0)}\int_{X\times\mathbb{D}^*/\mathbb{D}^*}(\sqrt{-1}\partial\bar\partial(r^2e^\varphi))^{n+1}e^{-r_\varphi^2}\\ &= -\frac{1}{(2\pi)^n\mathrm{vol}(\xi_0)}\int_{Y\times\mathbb{D}^*/\mathbb{D}^*}(d\chi+\sqrt{-1}\partial\bar\partial\varphi)^n\wedge\chi.\end{aligned}$$

In particular, $E(\varphi_t)$ is concave in $-\log|t|^2$.

Proof. The proof of the first identity is the same as the proof as in [LX17, Lemma 5.10]. The second identity follows from the first one and using the following identity on $X \times \mathbb{D}^*$ to calculate:

$$\sqrt{-1}\partial\bar{\partial}r_\varphi^2 = r_\varphi^2(d\chi + \sqrt{-1}\partial\bar{\partial}\varphi) + dr_\varphi^2 \wedge (\chi - \frac{1}{2}Jd\varphi).$$

□

Now assume that $(\mathcal{X}, \xi_0; \eta)$ is a $\mathbb{Q}$-Gorenstein test configuration of X. Because η commutes with ξ_0 and generates a $\mathbb{C}^*$-action, we can assume that $\mathcal{X}$ is embedded into $\mathbb{C}^N \times \mathbb{C}$ and the embedding is equivariant with respect to the $T \times \mathbb{C}^*$-action generated by $\{\xi_0, \eta\}$. If we write $\eta = \sum_i b_i z_i \frac{\partial}{\partial z_i}$ with $b_i \in \mathbb{Z}$ and let $\sigma(t) : \mathbb{C}^* \to GL(N, \mathbb{C})$ be the one-parameter subgroup generated by the vector field η. Then $\sigma(t)(z_i) = t^{b_i} z_i$ and we let

$$r(t)^2 := \sigma(t)^*(r^2) =: r^2 e^{\tilde{\varphi}(t)}.$$

The asymptotic of $E(\tilde{\varphi}_t)$ can be easily calculated:

Proposition A.10 (see [LX17, Proposition 5.13])**.** *We have the following identity:*

$$\lim_{t\to 0} \frac{E(\tilde{\varphi}_t)}{-\log|t|^2} = \frac{D_{-\eta}\mathrm{vol}(\xi_0)}{\mathrm{vol}(\xi_0)}. \tag{34}$$

Proof. We refer to [LX17] for details. Here we just sketch the key ingredients. Let $\xi_\epsilon = \xi - \epsilon\eta = \sum_i (a_i - \epsilon b_i) z_i \frac{\partial}{\partial z_i}$ and r_ϵ be a radius function for ξ_ϵ. Then we have:

$$\mathrm{vol}(\xi_\epsilon) = \frac{1}{n!(2\pi)^n} \int_{X_0} e^{-r_\epsilon^2} (\sqrt{-1}\partial\bar{\partial}r_\epsilon^2)^n. \tag{35}$$

Taking derivative with respect to ϵ in the above volume formula, we can derive:

$$D_{-\eta}\mathrm{vol}(\xi_0) \quad = \quad \frac{1}{(2\pi)^n (n-1)!} \int_{X_0} \theta e^{-r^2} (\sqrt{-1}\partial\bar{\partial}r^2)^n,$$

where we have denoted $\theta := \eta(\log r^2)$. We can then calculate (see [MSY08, Appendix C] or [LX17, Lemma 5.11]):

$$\begin{aligned} \frac{d}{d(-\log|t|^2)} E(\tilde{\varphi}_t) &= \frac{1}{(n-1)!(2\pi)^n \mathrm{vol}(\xi_0)} \int_X \dot{\tilde{\varphi}} e^{-r(t)^2} (\sqrt{-1}\partial\bar{\partial}r(t)^2)^n \\ &= \frac{1}{(n-1)!(2\pi)^n \mathrm{vol}(\xi_0)} \int_X \sigma(t)^*(\theta) e^{-\sigma^* r^2} \sigma^*(\sqrt{-1}\partial\bar{\partial}r^2)^n \\ &= \frac{1}{(n-1)!(2\pi)^n \mathrm{vol}(\xi_0)} \int_{X_t} \theta e^{-r^2} (\sqrt{-1}\partial\bar{\partial}r^2)^n. \end{aligned}$$

As explained in [LX17, Proof of Proposition 5.12], the last expression converges as $t \to 0$ to $D_{-\eta}\mathrm{vol}(\xi_0)/\mathrm{vol}(\xi_0)$.

By Proposition A.9 $E(\tilde{\varphi}_t)$ is concave in $-\log|t|^2$. So the statement follows from the above discussion and the following identity for concave functions:

$$\lim_{t\to 0} \frac{d}{d(-\log|t|^2)} E(\tilde{\varphi}_t) = \lim_{t\to 0} \frac{E(\tilde{\varphi}_t)}{-\log|t|^2}$$

□

We need the following basic result from [DS17] which generalizes Berndtsson's result to the Kähler cone setting.

Theorem A.11 ([DS17], see also [BBEGZ11, Ber15])**.** *Let $\varphi(x,t) = \varphi(x,|t|) : X \times \mathbb{D}^* \to \mathbb{R}$ be an upper semicontinuous function such that $\varphi_t := \varphi(\cdot, t) \in PSH(X, \xi_0)$ for each $t \in \mathbb{D}^*$. Assume $\sqrt{-1}\partial\bar{\partial}(r^2 e^\varphi) \geq 0$ over $X \times \mathbb{D}^*$ in the sense of currents. Then $G(\varphi_t)$ is convex in $-\log|t|^2$. Moreover, if $G(\varphi_t)$ is affine in $-\log|t|^2$, then there exists a holomorphic vector field η_0 on X commuting with ξ such that $r_{\varphi_t} = \sigma_t^* r_{\varphi_0}$ where $\sigma_t = \exp(\log|t| \cdot \eta_0)$.*

Let $(\mathcal{X}, \xi_0; \eta)$ be a $\mathbb{Q}$-Gorenstein test configuration of X with the projection map $\pi : \mathcal{X} \to \mathbb{C}$. Let $X_t := \pi^{-1}(t)$ be the fiber over $\{t\}$ and o_t the vertex point of X_t. Denote $\mathcal{X}^\circ = \mathcal{X} \setminus \{o_t; t \in \mathbb{C}\}$. In the following discussion, we denote by R^2 the function obtained by restricting r^2, considered as a function on $\mathbb{C}^N \times \mathbb{C}$, to $\mathcal{X}$ via a fixed the equivariant embedding $\mathcal{X} \to \mathbb{C}^N \times \mathbb{C}$: $R^2 = r^2\big|_{\mathcal{X}}$.

Definition A.12. *Denote by $PSH(\mathcal{X}|_{\mathbb{D}}, \xi_0)$ the set of bounded real functions Φ on $\mathcal{X}^\circ|_{\mathbb{D}}$ that satisfies:*

(1) $\tau \circ \Phi = \Phi$ for any $\tau \in T$;

(2) $R^2_\Phi := R^2 e^\Phi$ is a proper plurisubharmonic function on $\mathcal{X}|_{\mathbb{D}}$.

As before, we can treat functions in $PSH(\mathcal{X}|_{\mathbb{D}}, \xi_0)$ as transversal Kähler potentials on $\mathcal{X}|_{\mathbb{D}}$. If we also denote by χ the restriction of $\chi = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)\log R^2 = -\frac{1}{2} J d \log R^2$ to $\mathcal{Y} := \{R = 1\} \cap \mathcal{X}$, then we can similarly define $PSH(\mathcal{Y}, \xi_0)$ as Definition A.6.

Moreover, using the equivariant isomorphism $\iota : \mathcal{X}|_{\mathbb{D}^*} \cong X \times \mathbb{D}^*$, we can associate to any $\Phi \in PSH(\mathcal{X}|_{\mathbb{D}})$ plurisubharmonic function φ on $X \times \mathbb{D}^*$ and hence a path $\varphi_t \in PSH(X, \xi_0)$ such that $R^2_\Phi = \iota^*(r^2_\varphi)$. As an example, the path asssociated to $\Phi = 0$ and is given by $\tilde{\varphi}_t$.

Proposition A.13. *Assume $\Phi \in PSH(\mathcal{X}, \xi_0)$ and let $\varphi_t \in PSH(X, \xi_0)$ be the associated path. Then $G(\varphi_t)$ is subharmonic in t and its Lelong number at $t = 0$ is given by $1 - \mathrm{lct}(\mathcal{X}, \mathcal{X}_0)$.*

Proof. Since $R^2_\Phi = \iota^*(r^2 e^\varphi)$ is plurisubharmonic over $\mathcal{X}|_{\mathbb{D}^*} \cong X \times \mathbb{D}^*$. Applying Theorem A.11, we get $G(\varphi_t)$ is subharmonic in t. To see that it's subharmonic over $\mathbb{D}$, we just need to show that $G(\varphi_t)$ is uniformly bounded from above. Because Φ bounded, we know that

$$|G(\varphi_t) - G(\tilde{\varphi}_t)| \le C.$$

So we just need to show that $G(\tilde{\varphi}_t)$ is uniformly bounded from above.

Because η preserves the global section $s \in |mK_{\mathcal{X}}|$: $\mathcal{L}_\eta s = 0$. As a consequence, $dV_{X_t} = \left(\sqrt{-1}^{mn^2} s \wedge \bar{s}\right)^{1/m}\Big|_{X_t}$ satisfies $\sigma_t^* dV_{X_t} = dV_{X_1} = dV_X$. So we have:

$$G(\tilde{\varphi}_t) = -\log\left(\int_X e^{-\sigma(t)^* r^2} (\sigma^* dV_{X_t})\right) = -\log\left(\int_{X_t} e^{-r^2} dV_{X_t}\right).$$

Because $\mathcal{L}_{r\partial_r} dV_{X_t} = 2n dV_{X_t}$, we can write $dV_{X_t} = 2r^{2n-1} dr \wedge \Omega_{Y_t}$ and calculate:

$$\begin{aligned} \int_{X_t} e^{-r^2} dV_{X_t} &= (n-1)! \int_{Y_t} \Omega_Y \\ &= C_n \cdot \int_{\{r \le 1\} \cap X_t} e^{-r^2} dV_{X_t} \le C_n \int_{\{r \le 1\} \cap X_t} dV_{X_t}, \end{aligned} \tag{36}$$

where $C_n = \frac{(n-1)!}{\int_0^1 e^{-r^2} r^{2n-1} dr^2}$.

Now the upper boundedness of $G(\tilde{\varphi}_t)$ can be seen in two ways. For one way, one can resolve the singularity of $\{r \le 1\} \cap (\mathcal{X}|_{\mathbb{D}})$ and estimate the integral using the method as in [Li17a, Proof of Lemma 3.7] or [BJ17]. The other approximation approach is the following. Recall that r^2 is the radius function associated to the vector field $\xi_0 = \sum_i a_i z_i \frac{\partial}{\partial z_i}$. Now we choose a sequence of vector fields $\xi^{(k)} = \sum_i a_i^{(k)} z_i \frac{\partial}{\partial z_i}$ with $a_i^{(k)} \in \mathbb{Q}$ and $a_i^{(k)} \to a_i$ as $k \to +\infty$. Choose a sequence of new radius function $r^{(k)} = r_{\xi^{(k)}}$ such that $r^{(k)}$ is uniformly C^0-comparable to the functions $\sum_{i=1}^N |z_i|^{2/(a_i^{(k)})}$. Then there exist $C_1, C_2 > 0$ such that, for any $\epsilon > 0$, we have: $C_1 (r^{(k)})^{1-\epsilon} \le r \le C_2 (r^{(k)})^{1+\epsilon}$ for $k \gg 1$. So we get:

$$\int_{\{r \le 1\} \cap X_t} dV_{X_t} \le \int_{\left\{r^{(k)} \le C_1^{(1-\epsilon)^{-1}}\right\} \cap X_t} dV_{X_t}.$$

Because $a_i^{(k)}$ is rational, we can taking quotient of $\mathcal{X}$ by the $\mathbb{C}^*$-action generated by $\xi^{(k)} = \sum_i a_i^{(k)} \partial_{z^i}$ and reduces to the log Fano case considered in [Ber15] in which case the upper boundedness of $G(\tilde{\varphi}_t)$ was shown.

Finally we need to calculate the Lelong number of $G(\tilde{\varphi}_t)$ with respect to t. According to [Ber15], the Lelong number of $G(t)$ is equal to the infimum of c such that

$$\int_U e^{-G-(1-c)\log|t|^2} id\tau\wedge d\bar{\tau} = \int_{\mathcal{X}|_{\mathbb{D}}} e^{-r^2-(1-c)\log|t|^2} dV_{\mathcal{X}} < +\infty.$$

We have the following identity:

$$\int_{\mathcal{X}|_{\mathbb{D}}} e^{-r^2-(1-c)\log|\tau|^2} dV_{\mathcal{X}} = C_n \cdot \int_{\mathcal{X}|_{\mathbb{D}}\cap\{r\le 1\}} e^{-r^2-(1-c)\log|\tau|^2} dV_{\mathcal{X}}. \tag{37}$$

Because $e^{-1}\le e^{-r^2}\le 1$ is a bounded function, the right-hand-side of (37) is integrable if and only if $1-c<\mathrm{lct}(\mathcal{X}\cap\{r\le 1\}, \mathcal{X}_0\cap\{r\le 1\})$. Using the rescaling symmetry as used in (36), we see that $\mathrm{lct}(\mathcal{X}\cap\{r\le 1\}, \mathcal{X}_0\cap\{r\le 1\}) = \mathrm{lct}(\mathcal{X}, \mathcal{X}_0)$. So we are done.

□

Assume $r^2 e^{\varphi_{\mathrm{KE}}}$ with $\varphi_{\mathrm{KE}}\in PSH(X,\xi_0)$ is a radius function of a Ricci-flat Kähler cone metric on (X,ξ_0). Let $(\mathcal{X},\xi_0;\eta)$ be a test configuration of (X,ξ_0). We construct geodesic ray associated to $(\mathcal{X},\xi_0;\eta)$ by solving the homogeneous Monge-Ampère equation:

$$(\sqrt{-1}\partial\bar{\partial}(R^2 e^{\Phi}))^{n+1} = 0 \text{ on } \mathcal{X}|_{\mathbb{D}}, \quad \Phi|_{X\times S^1} = \varphi_{\mathrm{KE}}. \tag{38}$$

Using transversal point of view, this equation is equivalent to the following equation:

$$(d\chi+\sqrt{-1}\partial\bar{\partial}\Phi)^n\wedge\chi = 0 \text{ on } \mathcal{Y}|_{\mathbb{D}}, \quad \Phi|_{Y\times S^1} = \varphi_{\mathrm{KE}}|_Y. \tag{39}$$

By considering the envelope (or its equivalent formulation on $\mathcal{X}|_{\mathbb{D}}$)

$$\Phi := \sup\left\{\Psi\in PSH(\mathcal{Y}|_{\mathbb{D}},\xi_0) : \Psi\le \varphi_{\mathrm{KE}}|_Y \text{ on } \partial(\mathcal{Y}|_{\mathbb{D}}) = Y\times S^1\right\},$$

then the following result can be proved in exactly the same way as in [Ber15, Proposition 2.7] by using the transversal Kähler structures of $(\mathcal{Y},\xi_0)$. Note that this kind of extension has also been used in [DS17] (see also [CS15, HL18]).

Proposition A.14 (see [Ber15, Proposition 2.7])**.** *Φ is locally bounded such that R^2e^{Φ} has positive curvature current and satisfies $(\sqrt{-1}\partial\bar{\partial}(R^2e^{\Phi}))^{n+1}=0$ on $\mathcal{X}|_{\mathbb{D}}$.*

Finally we can give the proof of Theorem A.2.

Proof of Theorem A.2. Let Φ be the geodesic ray emanating from φ_{KE} that is determined by $(\mathcal{X},\xi_0)$. Let φ_t be the associated path in $PSH(X,\xi_0)$. Then because $(\sqrt{-1}\partial\bar{\partial}(R^2e^{\Phi}))^{n+1} = 0$, $E(\varphi_t)$ is affine in t by Proposition A.9. $G(\varphi_t)$ is subharmonic in t by Proposition A.13. So $D(t) := D(\varphi_t)$ is subharmonic over $\mathbb{D}$. Because $D(t)$ depends only $|t|$, $D(t)$ is convex in $-\log|t|^2$. Because $D(\varphi_t)\ge D(\varphi_{\mathrm{KE}})$ for any $t\in\mathbb{D}$, we see that $D(t)$ is a non-decreasing function in $-\log|t|^2$.

By Proposition A.10 and Proposition A.13, we have:

$$\lim_{t\to 0}\frac{D(t)}{-\log|t|^2} = \frac{D_{-\eta}\mathrm{vol}(\xi_0)}{\mathrm{vol}(\xi_0)} - (1-\mathrm{lct}(\mathcal{X},\mathcal{X}_0)) = D^{\mathrm{NA}}(\mathcal{X},\xi_0;\eta).$$

If $D^{\mathrm{NA}}(\mathcal{X},\xi_0;\eta)=0$, then because $D(t)$ is convex and non-decreasing in $-\log|t|^2$, we see that $D(t)$ is affine and hence $G(\varphi_t)$ is affine. So by Theorem A.11, there exists holomorphic vector field η_0 such that $\varphi_t = (\sigma_t)^*\varphi_{\mathrm{KE}}$ where $\sigma_t = \exp(\log|t|\eta_0)$. The rest of the argument is the same as [Ber15, Proposition 3.3] as extended to the Ricci-flat cone setting in [CS15].

□

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