# $\mathbb{G}$-uniform stability and Kähler-Einstein metrics on Fano varieties

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**Abstract**

Let $X$ be any $\mathbb{Q}$-Fano variety and $\mathrm{Aut}(X)_0$ be the identity component of the automorphism group of $X$. Let $\mathbb{G}$ denote a connected reductive subgroup of $\mathrm{Aut}(X)_0$. We prove that if $X$ is $\mathbb{G}$-uniformly K-stable, then it admits a Kähler-Einstein metric. The converse of this result holds true if $\mathbb{G}$ contains a maximal torus of $\mathrm{Aut}(X)_0$. These results give versions of Yau-Tian-Donaldson conjecture for arbitrary singular Fano varieties. A key new ingredient is a valuative criterion for the $\mathbb{G}$-uniform K-stability.

## Contents

# 1 Introduction

In this paper, a log Fano pair $(X, D)$ is a normal projective variety $X$ together with an effective $\mathbb{Q}$-Weil divisor $D$ such that $L := -(K_X + D)$ is an ample $\mathbb{Q}$-Cartier divisor and $(X, D)$ has at worst klt singularities. If $D = 0$, then $X$ is called a $\mathbb{Q}$-Fano variety. In [46], the author together with G. Tian and F. Wang proved the uniform version of Yau-Tian-Donaldson conjecture: a $\mathbb{Q}$-Fano variety $X$ with a discrete automorphism group admits a Kähler-Einstein metric if and only if $X$ is uniformly K-stable, if and only if $X$ is uniformly Ding-stable.

In this paper, we consider the case when the automorphism group is not discrete. In this case, Hisamoto [39] introduced $\mathbb{G}$-uniform stability condition (he called it relatively uniform stability for $\mathbb{G}$) and made an insightful observation that this stability condition corresponds nicely with an analytic criterion for equivariant properness which he obtained by using Darvas-Rubinstein's principle. Since we will use such type of analytic criterion to get Kähler-Einstein metric, Hisamoto's stability condition will play a basic role in our work.

**Notation:** In this paper, we will use the following notations:

(i) $\mathrm{Aut}(X, D)$ denotes the automorphism group of $(X, D)$ (i.e. the automorphism of $X$ that preserves $D$). $\mathrm{Aut}(X, D)_0$ is its identity component.

(ii) $\mathbb{G}$ is a connected reductive subgroup of $\mathrm{Aut}(X, D)_0$. $C(\mathbb{G})$ is the center of $\mathbb{G}$ and $\mathbb{T} := C(\mathbb{G})_0$ is the identity component of $C(\mathbb{G})$. We have $\mathbb{T} \cong (\mathbb{C}^*)^r = (S^1)^{\mathbb{C}}$.

(iii) $\mathbb{K}$ is a maximal compact subgroup of $\mathbb{G}$ that contains $(S^1)^r$.

**Definition 1.1** (see [39, 40])**.** *With the above notations, $(X, D)$ is called $\mathbb{G}$-uniformly K-stable if $\mathbb{G}$ is reductive and there exists $\gamma > 0$ such that for any $\mathbb{G}$-equivariant test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ of $(X, D, -(K_X + D))$, the following inequality holds true:*

$$\mathrm{CM}(\mathcal{X}, \mathcal{D}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}). \tag{1}$$

*See (44) for the definition of* CM *and (133) for $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}$. If one replace the* CM *invariant by $\mathbf{D}^{\mathrm{NA}}$ (see (46)), then one defines the $\mathbb{G}$-uniform Ding-stability of $(X, D)$ (called relatively uniform $\mathbf{D}$-stability for $\mathbb{G}$ in [39]).*

We will prove the following general existence result:

**Theorem 1.2.** *Let $(X, D)$ be a log Fano pair. $(X, D)$ admits a Kähler-Einstein metric if $(X, D)$ is $\mathbb{G}$-uniformly K-stable, or equivalently if $(X, D)$ is $\mathbb{G}$-uniformly Ding-stable.*

In the case when $X$ is a smooth Fano manifold and $D = \emptyset$, the above result can be derived from the work [21] (see Remark 1.6), which depends on the method of partial $C^0$-estimates. Again in the smooth case, a different argument for the statement involving only Ding-stability, which depends on Berman-Boucksom-Jonsson's variational approach, is also claimed by Hisamoto in [40] (however see Remark 5.10). Here we don't require extra constraint on the singularities of $(X, D)$.

To prove Theorem 1.2, we first need to derive a valuative criterion for $\mathbb{G}$-uniform Ding/K-stability. To state this criterion, first note that by the reductivity of $\mathbb{G}$, $\mathbb{T} = C(\mathbb{G})_0$ is isomorphic to a complex torus $(\mathbb{C}^*)^r$. Set

$$N_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}), \quad N_{\mathbb{Q}} = N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N_{\mathbb{R}} = N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2}$$

Also set $M_{\mathbb{Z}} = N_{\mathbb{Z}}^{\vee}, M_{\mathbb{Q}} = M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, M_{\mathbb{R}} = M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

Denote by $\mathrm{Val}(X)$ the set of (real) valuations on $X$. For any valuation $v \in \mathrm{Val}(X)$, denote by $A_{(X,D)}(v)$ the log discrepancy of $v$ and by $\mathring{\mathrm{Val}}(X)$ the set of valuations $v$ satisfying $A_{(X,D)}(v) < +\infty$. Then $\mathring{\mathrm{Val}}(X)$ contains the set $X^{\mathrm{div}}_{\mathbb{Q}}$ of all divisorial valuations which are of the form $\lambda \cdot \mathrm{ord}_E$ where $\lambda > 0 \in \mathbb{Q}$. Denote by $\mathrm{Val}(X)^{\mathbb{T}}$ (resp. $\mathrm{Val}(X)^{\mathbb{G}}$) the set of $\mathbb{T}$-invariant (resp. $\mathbb{G}$-invariant) valuations on $X$. Then $\mathring{\mathrm{Val}}(X)^{\mathbb{T}} := \mathrm{Val}(X)^{\mathbb{T}} \cap \mathring{\mathrm{Val}}(X)$ (resp. $\mathring{\mathrm{Val}}(X)^{\mathbb{G}} = \mathrm{Val}(X)^{\mathbb{G}} \cap \mathring{\mathrm{Val}}(X)$) denotes the set of $\mathbb{T}$-invariant (resp. $\mathbb{G}$-invariant) valuations on $X$ satisfying $A_{(X,D)}(v) < +\infty$. We observe that $N_{\mathbb{R}}$ acts on $(\mathring{\mathrm{Val}}(X))^{\mathbb{T}}$: $(\xi, v) \mapsto v_\xi$ (see section 2.3). If we choose any $\ell_0$ such that $-\ell_0(K_X+D)$ is Cartier, then $v$ induces a filtration $\mathcal{F}_v = \mathcal{F}_v R_\bullet$ on $R := R^{(\ell_0)} := \bigoplus_{m=0}^{+\infty} H^0(X, -m\ell_0(K_X+D))$ (see (67)). Define an invariant (see (68)):

$$S_{-(K_X+D)}(v) := \frac{1}{\ell_0^n(-(K_X+D))^{\cdot n}} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}_v^{(x)}\right) dx. \tag{3}$$

Given $(X, D)$, this is an invariant of $v$ and does not depend on the choice of $\ell_0$.

Let $\mathfrak{t}$ denote the Lie algebra of $(S^1)^r \subset \mathbb{T}$ which is identified with the set of holomorphic fields generated by the elements of $\mathfrak{t}$. Note that there is a natural isomorphism $\mathfrak{t} \cong N_{\mathbb{R}}$.

**Theorem 1.3.** *With the above notations, the following statements are equivalent:*

*(1)* $(X, D)$ *is* $\mathbb{G}$*-uniformly K-stable;*

*(2)* $(X, D)$ *is* $\mathbb{G}$*-uniformly Ding-stable;*

*(3)* $\mathbb{G}$ *is reductive,* $\mathrm{Fut} \equiv 0$ *on* $N_{\mathbb{R}}$ *and there exists* $\delta_{\mathbb{G}} > 1$ *such that for any* $\mathbb{G}$*-invariant divisorial valuation* $v$ *over* $X$ *there exists* $\xi \in N_{\mathbb{R}}$ *satisfying* $A_{(X,D)}(v_\xi) \geq \delta_{\mathbb{G}} \cdot S_{-(K_X+D)}(v_\xi)$.

*(4)* $\mathbb{G}$ *is reductive,* $\mathrm{Fut} \equiv 0$ *on* $N_{\mathbb{R}}$ *and there exists* $\delta_{\mathbb{G}} > 1$ *such that for any* $v \in \mathring{\mathrm{Val}}(X)^{\mathbb{G}}$ *there exists* $\xi \in N_{\mathbb{R}}$ *satsifying* $A_{(X,D)}(v_\xi) \geq \delta_{\mathbb{G}} \cdot S_{-(K_X+D)}(v_\xi)$.

*(5)* $(X, D)$ *is* $\mathbb{G}$*-uniformly K-stable among* $\mathbb{G}$*-equivariant special test configurations.*

Here the last condition (5) means that in Definition 1.1 the inequality (1) is required only for $\mathbb{G}$-equivariant special test configurations (see Definition 2.16 and 3.1).

In practice, we have the following result that serves the same purpose as what a result from [21] does for obtaining Kähler-Einstein metrics on varieties with large symmetries. Again the advantage of our result is that it works for all singular Fano varieties.

**Corollary 1.4.** *Assume that there are only finitely many* $\mathbb{G}$*-equivariant special degenerations of* $(X, D)$*. If* $(X, D)$ *is* $\mathbb{G}$*-equivariantly K-polystable, then* $(X, D)$ *is* $\mathbb{G}$*-uniformly K-stable. Hence* $(X, D)$ *admits a Kähler-Einstein metric.*

Here by a $\mathbb{G}$-equivariant special degeneration we mean a special test configuration but without the data $\eta$ that generates the $\mathbb{C}^*$-action.

We will then show that the converse to Theorem 1.2 holds true if $\mathbb{G}$ contains a maximal torus of $\mathrm{Aut}(X)_0$. This is true because the existence of Kähler-Einstein metrics implies a properness condition involving such $\mathbb{G}$, which we prove by combining the works of Darvas-Rubinstein and Hisamoto, together with some properties of reductive groups proved in Appendix A. So we get the sufficient and necessary algebraic conditions for the existence of Kähler-Einstein metrics for any (singular) Fano variety.

**Theorem 1.5.** *Let* $(X, D)$ *be a log Fano pair. Then* $(X, D)$ *admits a Kähler-Einstein metric if and only if* $\mathrm{Aut}(X, D)_0$ *is reductive and* $(X, D)$ *is* $\mathbb{G}$*-uniformly Ding-stable, where* $\mathbb{G}$ *is any connected reductive group of* $\mathrm{Aut}(X, D)_0$ *that contains a maximal torus of* $\mathrm{Aut}(X, D)_0$.

Theorem 1.5 is the first versions of Yau-Tian-Donaldson conjecture for arbitrary Fano varieties. We make some remarks about the above results.

**Remark 1.6.** *In this remark we use the definition 3.19 and Remark 3.20.*

1. *By definitions,* $\mathbb{G}$*-equivariantly uniform K-stability implies* $\mathbb{G}$*-uniform K-stability (since* $\mathbf{J}^{\mathrm{NA}} \geq \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}$*). The converse is not true in general. In fact, it is easy to show that* $\mathbb{G}$*-equivariantly uniform K-stability is equivalent to two conditions together:* $\mathbb{G}$*-uniform K-stability plus the center* $C(\mathbb{G})$ *being discrete. So for the above results, if* $C(\mathbb{G})$ *is*

*discrete, we can replace the $\mathbb{G}$-uniform K-stability (Ding-stability) by $\mathbb{G}$-equivariantly uniform K-stability (Ding-stability). We note that $\mathbb{G}$-equivariantly uniform K-stability was considered recently in [36] and [56].*

2. *It can be shown that $\mathbb{G}$-uniform K-stability implies $\mathbb{G}$-equivariant K-polystability (Lemma 3.21). Conversely $\mathbb{G}$-equivariant K-polystability does not in general imply $\mathbb{G}$-uniform K-stability if $\mathbb{G}$ is too small compared to $\mathrm{Aut}(X, D)_0$ (e.g. take $X = \mathbb{P}^n$ and $\mathbb{G} = \{e\}$). With our result, it is natural to expect that for any $\mathbb{G}$ containing a maximal torus, $\mathbb{G}$-equivariant K-polystability (or just K-polystability) is equivalent to $\mathbb{G}$-uniform K-stability (see also [50]). This is known in the smooth case by the works in [21] and [39] through the existence of Kähler-Einstein metrics.*
3. *The connectedness assumption of $\mathbb{G}$ is only essentially used in the proof of the implication from K-stability to Ding-stability, where we used the $\mathbb{C}^* \times \mathbb{G}$-equivariant MMP process to get special test configurations where K-stability coincides with Ding-stability. The existence part of the proof still goes through and hence a version of Yau-Tian-Donaldson conjecture via Ding-stability holds true for more general case of disconnected subgroups of $\mathrm{Aut}(X)$ by probably invoking a more general version of analytic criterion than Theorem 2.15. We leave to the reader to write down the valid statement.*

We end the introduction with a short discussion of proofs. The general idea for the proof of Theorem 1.3 parallels the idea for the proof of valuative criterion by Fujita and the author in [32, 43], which uses the equivariantly relative MMP process from [44] (see also section 4.2). However, we need to understand in detail how to relate the twists of valuations to the twists of non-Archimedean metrics including those from test configurations. Note that the notion of twist of test configurations appeared in Hisamoto's work [38, 39]. We also need to establish that the $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}$ energy for filtration (associated to valuations) can be approximated by $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}$ for test configurations. The other observation is that the calculations for the decreasing of $\mathbf{D}^{\mathrm{NA}} - \epsilon\mathbf{J}^{\mathrm{NA}}$ (for $\epsilon \in [0, 1]$) in [32] are compatible with twists.

In addition to the valuative criterion in Theorem 1.3, the work here is a synthesis of ideas from [8], [39] and [46], and further carries out Berman-Boucksom-Jonsson's program of variational approach (proposed in [7, 8]) to Yau-Tian-Donaldson conjecture for all $\mathbb{Q}$-Fano varieties. However compared with all these previous works, we need to find new ways to deal with difficulties arising from singularities and continuous automorphism groups. To overcome the difficulties caused by singularities, we use the perturbative idea from our previous work ([45, 46]). But we will not directly prove $\mathbb{G}$-uniform stability on the resolution as in these works. Instead, we need to work with valuations that approximately calculate the $\mathbf{L}^{\mathrm{NA}}$ part of the non-Archimedean Ding energy. This will also allow us to effectively use a key identity (see (116) and (123)) about twists of non-Archimedean metrics in order to deal with the case with continuous automorphism groups. In addition, our proof depends on monotonicity of *both parts* of the $\mathbf{J}$ energy functional and some delicate uniform estimates of non-Archimedean quantities. The main line of arguments is essentially contained in a long chain of (in)equalities in section 5.4. In particular our way to overcome difficulties caused by continuous automorphism groups is quite different with Hisamoto's argument (see Remark 5.10).

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# 2 Preliminaries

## 2.1 Space of Kähler metrics over singular projective varieties

Let $Z$ be an $n$-dimensional normal projective variety and $Q$ a Weil divisor that is not necessarily effective. Assume that $L$ is an ample $\mathbb{Q}$-Cartier divisor. Choose a smooth Hermitian metric $e^{-\psi}$ on $L$ with a smooth semi-positive curvature form $\omega = \sqrt{-1}\partial\bar{\partial}\psi \in 2\pi c_1(L)$.

We will use the following spaces:

$$\mathrm{PSH}(\omega) := \mathrm{PSH}(Z,\omega) = \left\{ \text{ u.s.c. function } u \in L^1_{\mathrm{loc}}(Z); \quad \omega_u := \omega + \sqrt{-1}\partial\bar{\partial}u \geq 0 \right\}; \tag{4}$$

$$\mathcal{H}(\omega) := \mathcal{H}(Z,\omega) = \mathrm{PSH}(\omega) \cap C^\infty(Z); \tag{5}$$

$$\mathrm{PSH}_{\mathrm{bd}}(\omega) := \mathrm{PSH}_{\mathrm{bd}}(Z,\omega) = \mathrm{PSH}(\omega) \cap \{\text{bounded functions on } Z\}; \tag{6}$$

$$\mathrm{PSH}(L) := \mathrm{PSH}([\omega]) := \{\varphi = \psi + u; u \in \mathrm{PSH}(\omega)\}; \tag{7}$$

$$\mathrm{PSH}_{\mathrm{bd}}(L) := \mathrm{PSH}_{\mathrm{bd}}([\omega]) := \{\varphi = \psi + u; u \in \mathrm{PSH}_{\mathrm{bd}}(\omega)\}. \tag{8}$$

Note that $\mathrm{PSH}([\omega])$ is equal to the space of positively curved (possibly singular) Hermitian metrics $\{e^{-\varphi} = e^{-\psi-u}\}$ on the $\mathbb{Q}$-line bundle $L$. Rigorously $\psi + u$ is not a globally defined function, but rather a collection of local psh functions that satisfy the obvious compatible condition with respect to the transition functions of the $\mathbb{Q}$-line bundle. However for the simplicity of notations, we will abuse this notation.

Note that we have weak topology on $\mathrm{PSH}(\omega)$ which coincides with the $L^1$-topology. If $u_j$ converges to $u$ weakly, then $\sup(u_j) \to \sup(u)$ by Hartogs' lemma for plurisubharmonic functions [35, Theorem 1.46]. Moreover, we have the following lemma, which in the smooth case can be proved by using Green's formula.

**Lemma 2.1.** *Let $u_j \in \mathrm{PSH}(\omega)$ be a sequence such that $\sup(u_j) = 0$. Then there exists a constant $C > 0$ independent of $j$, such that:*

$$\int_X u_j \omega^n \geq -C. \tag{9}$$

*Proof.* Indeed, by Hartogs lemma in [35, Theorem 1.46] (applied on a global resolution of $X$), we know that $u_j$ converges to $u_\infty \in PSH(\omega)$ and $\int_X u_j\omega^n \to \int_X u\omega^n > -\infty$. ☐

**Proposition 2.2** ([20, Corollary C]). *For any $u \in \mathrm{PSH}(Z,\omega)$ there exists a sequence of smooth functions $u_j \in \mathrm{PSH}(Z,\omega)$ which decrease pointwise on $Z$ so that $\lim_{j\to+\infty} u_j = u$ on $Z$.*

For any $u \in \mathrm{PSH}(Z,\omega)$, define:

$$\omega_u^n := \lim_{j\to+\infty} \mathbf{1}_{\{u>-j\}} \left(\omega + \sqrt{-1}\partial\bar{\partial}\max(u,-j)\right)^n. \tag{10}$$

We will use the space $\mathcal{E}^1$ of finite energy $\omega$-psh functions (see [37]):

$$\mathcal{E}(\omega) := \mathcal{E}(Z,\omega) = \left\{ u \in \mathrm{PSH}(Z,\omega); \int_Z \omega_u^n = \int_Z \omega^n \right\}; \tag{11}$$

$$\mathcal{E}^1(\omega) := \mathcal{E}^1(Z,\omega) = \left\{ u \in \mathcal{E}(Z,\omega); \int_Z |u|\omega_u^n < \infty \right\}; \tag{12}$$

$$\mathcal{E}^1(L) := \mathcal{E}^1(Z,L) = \left\{ \psi + u; u \in \mathcal{E}^1(Z,\omega) \right\}. \tag{13}$$

We have the inclusion $\mathrm{PSH}_{\mathrm{bd}}(\omega) \subset \mathcal{E}^1(\omega)$.

Set $V = L^{\cdot n}$. For any $\varphi \in \mathrm{PSH}([\omega])$ such that $\varphi - \psi \in \mathcal{E}^1(L)$, we have the following important functional:

$$\mathbf{E}(\varphi) \quad := \quad \mathbf{E}_\psi(\varphi) = \frac{1}{(n+1)(2\pi)^n V} \sum_{i=0}^n \int_Z (\varphi - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^{n-i} \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^i. \tag{14}$$

Following [6], we endow $\mathcal{E}^1$ with the strong topology.

**Definition 2.3.** *The strong topology on $\mathcal{E}^1$ is defined to as the coarsest refinement of the weak topology such that $\mathbf{E}$ is continuous.*

For any interval $I \subset \mathbb{R}$, denote the Riemann surface

$$\mathbb{D}_I = I \times S^1 = \{\tau \in \mathbb{C}^*; s = \log|\tau| \in I\}.$$

**Definition 2.4** (see [8, Definition 1.3])**.** *A $\omega$-psh path, or just the psh path, on an open interval $I$ is a map $U = \{u(s)\} : I \to \mathrm{PSH}(\omega)$ such that the $U(\cdot,\tau) := U(\log|\tau|)$ is a $p_1^*\omega$-psh function on $X \times \mathbb{D}_I$. A psh ray (emanating from $u_0$) is a psh path on $(0,+\infty)$ (with $\lim_{t\to 0} u(s) = u_0$). Note in the literature, psh path (resp. psh ray) are also called subgeodesic (resp. subgeodesic ray).*

*In the above situation, we also say that $\Phi(s) = \{\psi_0 + u(s)\}$ is a psh path (resp. a psh ray).*

We will use geodesics connecting bounded potentials.

**Proposition 2.5** ([28, Proposition 1.17])**.** *Let $u_0, u_1 \in \mathrm{PSH}_{\mathrm{bd}}(\omega)$. Then*

$$U = \sup\left\{u; u \in \mathrm{PSH}(Z \times \mathbb{D}_{[0,1]}, p_1^*\omega); \quad U \le u_{0,1} \text{ on } \partial(Z \times \mathbb{D}_{[0,1]})\right\}. \tag{15}$$

*is the unique bounded $\omega$-psh function on $Z \times \mathbb{D}_{[0,1]}$ that is the solution of the Dirichlet problem:*

$$(\omega + \sqrt{-1}\partial\bar{\partial}U)^{n+1} = 0 \text{ on } Z \times \mathbb{D}_{[0,1]}, \quad U|_{Z\times\partial\mathbb{D}_{[0,1]}} = u_{0,1}. \tag{16}$$

We will call $\Phi = \{\varphi(s) = \psi + U(\cdot,s)\}$ the geodesic segment joining $\varphi_0 = \psi + u_0$ and $\varphi_1 = \psi + u_1$.

For finite energy potentials $u_0, u_1 \in \mathcal{E}^1(\omega)$, let $u_0^j, u_1^j$ be bounded smooth $\omega$-psh functions decreasing to $u_0, u_1$ (see Proposition 2.2). Let $u_t^j$ be the bounded geodesic connecting $u_0^j$ to $u_1^j$. It follows from the maximum principle that $j \to u_t^j$ is non-increasing. Set:

$$u_t := \lim_{j\to+\infty} u_t^j. \tag{17}$$

Then $U = \{u_t\}$ is a finite-energy geodesic joining $u_0$ to $u_1$ as stated in the following result.

**Theorem 2.6** ([28, Proposition 4.6], [8, Theorem 1.7])**.** *For any $u_0, u_1 \in \mathcal{E}^1(\omega)$, the psh geodesic joining them exists, and defines a continuous map $U : [0,1] \to \mathcal{E}^1$ in the strong topology.*

Generalizing Darvas' result in the smooth case ([22]), the works in [24, 28] showed that $\mathcal{E}^1$ can be characterized as the metric completion of $\mathcal{H}(\omega)$ under a Finsler metric $d_1$ which can be defined as follows. Fix a log resolution $\mu : Y \to Z$ and a Kähler form $\omega_P > 0$ on $Y$. Then

$$\omega_\epsilon := \mu^*\omega + \epsilon\omega_P \tag{18}$$

is a Kähler form and one can define Darvas' Finsler metric $d_{1,\epsilon}$ on $\mathcal{H}(Z, \omega_\epsilon)$. Note that $u \in \mathcal{H}(Z, \omega)$ implies $u \in \mathcal{H}(Y, \omega_\epsilon)$. One then defines (see [28, Definition 1.10])

$$d_1(u_0, u_1) = \liminf_{\epsilon\to 0} d_{1,\epsilon}(u_0, u_1).$$

It is known that $u_j \to u$ in $\mathcal{E}^1$ under the strong topology if and only if $d_1(u_j, u) = 0$. Moreover in this case the Monge-Ampère measures $(\sqrt{-1}\partial\bar{\partial}(\psi + u_j))^n$ converges weakly to $(\sqrt{-1}\partial\bar{\partial}(\psi + u))^n$.

## 2.2 Analytic criterion for the existence of Kähler-Einstein metrics

For any $\varphi \in \mathrm{PSH}([\omega])$ such that $\varphi - \psi \in \mathcal{E}^1(\omega)$, we have the following well-studied functionals:

$$\begin{aligned}
\mathbf{E}(\varphi) &:= \mathbf{E}_\psi(\varphi) = \frac{1}{(n+1)(2\pi)^n V}\sum_{i=0}^{n}\int_Z (\varphi - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^{n-i} \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^i \quad (19)\\
\mathbf{\Lambda}(\varphi) &:= \mathbf{\Lambda}_\psi(\varphi) = \frac{1}{(2\pi)^n V}\int_Z (\varphi - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^n, \quad (20)\\
\mathbf{J}(\varphi) &:= J_\psi(\varphi) = \mathbf{\Lambda}_\psi(\varphi) - \mathbf{E}_\psi(\varphi)\\
&= \frac{1}{(2\pi)^n V}\int_Z (\varphi - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^n - \mathbf{E}_\psi(\varphi), \quad (21)\\
(\mathbf{I} - \mathbf{J})(\varphi) &:= (\mathbf{I} - \mathbf{J})_\psi(\varphi) = \mathbf{E}_\psi(\varphi) - \frac{1}{(2\pi)^n V}\int_Z (\varphi - \psi)(\sqrt{-1}\partial\bar{\partial}\varphi)^n. \quad (22)
\end{aligned}$$

A key property we will need is the monotonicity of $\mathbf{\Lambda}$ and $\mathbf{E}$ functionals:

$$\varphi_1 \le \varphi_2 \quad \Longrightarrow \quad \mathbf{\Lambda}(\varphi_1) \le \mathbf{\Lambda}(\varphi_2) \quad \textit{and} \quad \mathbf{E}(\varphi_1) \le \mathbf{E}(\varphi_2). \tag{23}$$

Let $\mu : Y \to Z$ be a log resolution of singularities such that $\mu^{-1}Z^{\mathrm{sing}} = \sum_k E_k$ is the reduced exceptional divisor, $Q' := \mu_*^{-1}Q$ is the strict transform of $Q$ and $Q' + \sum_k E_k$ has simple normal crossings. We can write:

$$K_Y + Q' = \mu^*(K_Z + Q) + \sum_k a_k E_k. \tag{24}$$

**Definition 2.7.** *$(Z, Q)$ is said to have sub-Klt singularities if there exists a log resolution of singularities as above such that $a_k > -1$ for all $k$. If $Q$ is moreover effective, then $(Z, Q)$ is said to have Klt singularities.*

Fix $\ell_0 \in \mathbb{N}^*$ such that $\ell_0(K_Z + Q)$ is Cartier. If $\sigma$ is a nowhere-vanishing holomorphic section of the corresponding line bundle over a smooth open set $U$ of $Z$, then there is a pull-back meromorphic volume form on $\mu^{-1}(U)$:

$$\mu^*\left(\sqrt{-1}^{\ell_0 n^2}\sigma \wedge \bar{\sigma}\right)^{1/\ell_0} = \prod_i |z_i|^{2a_i} dV, \tag{25}$$

where $\{z_i\}$ are local holomorphic coordinates and $dV$ is a smooth volume for on $Y$. If $(Z, Q)$ is sub-Klt, then the above volume form is locally integrable.

**Definition 2.8** (see [6, section 3]). *Assume $L = \lambda^{-1}(-K_Z - Q)$ is an ample $\mathbb{Q}$-line bundle for $\lambda > 0 \in \mathbb{Q}$. Let $\varphi \in \mathcal{E}^1(Z, L)$ be a finite energy Hermitian metric on the $\mathbb{Q}$-line bundle $L$. The adapted measure $\mathrm{mes}_\varphi$ is a globally defined measure:*

$$\frac{e^{-\lambda\varphi}}{|s_Q|^2} := \mathrm{mes}_\varphi = \left(\sqrt{-1}^{\ell_0 n^2} \sigma \wedge \bar{\sigma}\right)^{1/\ell_0} |\sigma^*|^{2/\ell_0}_{\ell_0 \lambda\varphi}, \tag{26}$$

*where $\sigma^*$ is the dual nowhere-vanishing section of $-\ell_0(K_Z + Q)$.*

The Ding- and Mabuchi- functionals on $\mathcal{E}^1(Z, L)$ are defined as follows:

$$\mathbf{L}(\varphi) = \mathbf{L}_{(Z,Q)}(\varphi) = -\log\left(\int_Y e^{-\varphi}\frac{1}{|s_Q|^2}\right) \tag{27}$$

$$\mathbf{D}(\varphi) = \mathbf{D}_{(Z,Q),\psi}(\varphi) = \mathbf{D}_\psi(\varphi) = -\mathbf{E}_\psi(\varphi) + \mathbf{L}_{(Z,Q)}(\varphi) \tag{28}$$

$$\mathbf{H}(\varphi) := \mathbf{H}_{(Z,Q),\psi}(\varphi) = \frac{1}{(2\pi)^n L^{\cdot n}} \int_X \log\frac{|s_Q|^2(\sqrt{-1}\partial\bar{\partial}\varphi)^n}{e^{-\psi}}(\sqrt{-1}\partial\bar{\partial}\varphi)^n \tag{29}$$

$$\mathbf{M}(\varphi) := \mathbf{M}_{(Z,Q),\psi}(\varphi) = \mathbf{M}_\psi(\varphi) = \mathbf{H}(\varphi) - (\mathbf{I} - \mathbf{J})_\psi(\varphi). \tag{30}$$

In the rest of this subsection, we will assume $(Z, Q) = (X, D)$ is a log Fano pair. In other words, we assume that $D$ is an effective divisor, $L = -K_X - D$ is an ample $\mathbb{Q}$-Cartier divisor and $(X, D)$ has klt singularities.

**Definition 2.9.** *A finite energy Hermitian metric $\varphi \in \mathcal{E}^1(X, -(K_X + D))$ is a Kähler-Einstein (Hermitian) metric on $(X, D)$ if it satisfies the following equation in the pluripotential sense:*

$$(\sqrt{-1}\partial\bar{\partial}\varphi)^n = \frac{e^{-\varphi}}{|s_D|^2}. \tag{31}$$

By [6], it is known that any Kähler-Einstein metric $\varphi$ is automatically bounded, smooth on $X^{\mathrm{reg}}$.

**Definition 2.10** ([6, Definition 1.3]). *A positive measure $\nu$ on $X$ is tame if $\mu$ puts no mass on closed analytic sets and if there is a resolution of singularities $\mu : Y \to X$ such that the lift $\nu_Y$ of $\nu$ to $Y$ has $L^p$ density for some $p > 1$.*

The following compactness result is very important in the variational approach to solving Monge-Ampère equations using the pluripotential theory.

**Theorem 2.11** ([6, Theorem 2.17]). *Let $\nu$ be a tame probability measure on $X$. For any $C > 0$, the following set is compact in the strong topology:*

$$\left\{u \in \mathcal{E}^1(X, \omega); \quad \sup_M u = 0, \quad \int_Z \log\frac{\omega_u^n}{\nu}\omega_u^n < C\right\}.$$

Let $\mathbb{G}$ be a connected reductive subgroup of $\mathrm{Aut}(X, D)_0$ and $\mathbb{T} := C(\mathbb{G})_0 \cong (\mathbb{C}^*)^r = ((S^1)^r)^{\mathbb{C}}$ be the identity component of the center $C(\mathbb{G})$. Any $\xi \in N_{\mathbb{R}}$ corresponds to a holomorphic vector field written as $\xi - iJ\xi$ where $J$ is the complex structure (on the regular part). In other words, we identify $\xi$ with a real vector field and $J\xi \in \mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of $(S^1)^r$. For any $\xi \in N_{\mathbb{R}}$, let $\sigma_\xi(\mathfrak{s}) : \mathbb{C} \to \mathbb{G}$ be the one parameter subgroup generated by $\xi$. Then we have:

$$\sigma_\xi(\mathfrak{s} = s + iu) = \exp(s\xi) \cdot \exp(uJ\xi). \tag{32}$$

If $\xi \in N_{\mathbb{Z}}$, then $\sigma_\xi \circ (-\log) =: \hat{\sigma}_\xi : \mathbb{C}^* \to \mathbb{G}$ is a well defined one parameter subgroup. In this paper, we will freely use the change of variables:

$$\mathbb{C}^* \to \mathbb{R}, \quad t \mapsto -\log|t| =: s. \tag{33}$$

Let $\mathbb{K}$ be a maximal compact subgroup of $\mathbb{G}$ containing $(S^1)^r$. Denote by $(\mathcal{E}^1)^{\mathbb{K}} := (\mathcal{E}^1(L))^{\mathbb{K}}$ the set of $\mathbb{K}$-invariant finite energy positively curved Hermitian metrics on $L$. For any $\varphi \in (\mathcal{E}^1)^{\mathbb{K}}$ define:

$$\mathbf{J}_{\mathbb{T}}(\varphi) := \mathbf{J}_{\psi,\mathbb{T}}(\varphi) := \inf_{\sigma\in\mathbb{T}} \mathbf{J}_{\psi}(\sigma^*\varphi). \tag{34}$$

**Lemma 2.12** ([39, Lemma 1.9]). *The function $\sigma \mapsto \mathbf{J}_\psi(\sigma^*\varphi)$ defined on $\mathbb{T} \cong N_{\mathbb{R}} \times (S^1)^r$ is $(S^1)^r$ invariant, convex and proper. As a consequence there always exists $\sigma \in \mathbb{T}$ that achieves the infimum.*

Indeed, by the $\mathbb{K}$-invariance (hence $(S^1)^r$-invariance), $\mathbf{J}_\psi(\sigma^*\varphi) = \mathbf{J}_\psi(\sigma_\xi(1)^*\varphi)$ can be seen as a function for $\xi \in N_{\mathbb{R}} \cong \mathbb{R}^r$. For convexity, see Proposition 5.1. To see its properness, note that the slope of this function along $\mathbb{R}_{>0}\xi$ for any $\xi \in N_{\mathbb{R}}$ is given by $\xi \mapsto \mathbf{J}'^{\infty}(\sigma_\xi(s)^*\psi) = \mathbf{J}^{\mathrm{NA}}(((X, -K_X) \times \mathbb{C})_\xi)$ which is strictly positive if $\xi \neq 0$ (see [14]).

**Definition 2.13** ([26, 39]). *We say that the energy $\mathbf{F} \in \{\mathbf{D}, \mathbf{M}\}$ is $\mathbb{G}$-proper (usually called coercive in the literature) if there exists $\gamma > 0$, $C > 0$ such that for any $\varphi \in (\mathcal{E}^1)^{\mathbb{K}}$ we have:*

$$\mathbf{F}(\varphi) \geq \gamma \cdot \mathbf{J}_{\mathbb{T}}(\varphi) - C. \tag{35}$$

**Theorem 2.14** ([6], [26], [24], [39, Theorem 3.4]). *Let $(X, D)$ be a log Fano pair. Let $\mathbb{G}$ be a connected reductive subgroup of* $\mathrm{Aut}(X, D)_0$*, and set $\mathbb{T} = C(\mathbb{G})_0$ and $\mathbb{K} \subset \mathbb{G}$ as before. Consider the following conditions:*

*(1) The Ding energy is $\mathbb{G}$-proper.*

*(2) The Mabuchi energy is $\mathbb{G}$-proper.*

*(3) $(X, D)$ admits a $\mathbb{K}$-invariant Kähler-Einstein metric.*

*Then condition (1) or (2) implies condition (3).*

*Moreover, if we assume that* $\mathrm{Aut}(X, D)_0$ *is reductive and set $\mathbb{G} = \mathrm{Aut}(X, D)_0$, then all of the above conditions are equivalent.*

The existence part of the above result can be derived from the work in [6, 39]. For the reader's convenience, we sketch the proof of (2) $\Rightarrow$ (3) and refer the details to [6, 24, 28]. Because Mabuchi energy is bigger than the Ding energy, (1) $\Rightarrow$ (3) also follows.

*Sketch of the proof of* (2) $\Rightarrow$ (3). Assume that $\mathbf{M}$ is $\mathbb{G}$-proper. Then $\mathbf{M}$ is bounded from below over $(\mathcal{E}^1)^{\mathbb{K}}$. Choose a sequence of potentials $\varphi_j \in (\mathcal{E}^1)^{\mathbb{K}}$ such that $\mathbf{M}(\varphi_j) \to \inf_{(\mathcal{E}^1)^{\mathbb{K}}} \mathbf{M}(\varphi)$. Then $\mathbf{J}_{\mathbb{T}}(\varphi_j) \leq C$ independent of $j$. By Lemma 2.12 there exists $\sigma_j \in \mathbb{T}$ such that $\tilde{\varphi}_j := \sigma_j^* \varphi_j$ satisfies $\mathbf{J}(\tilde{\varphi}_j) = \mathbf{J}_{\mathbb{T}}(\varphi_j)$. Clearly $\tilde{\varphi}_j \in (\mathcal{E}^1)^{\mathbb{K}}$. Moreover we can assume that $\sup(\tilde{\varphi}_j - \psi) = 0$.

From the $\mathbb{G}$-properness and using the fact that $\mathbf{M}$ is linear along one parameter group of $\mathbb{T}$ (with slope given by the Futaki invariant), we see that $\mathbf{M}$ is invariant under the $\mathbb{T}$-action on $(\mathcal{E}^1)^{\mathbb{K}}$. Moreover, use the expression $\mathbf{M} = \mathbf{H} - (\mathbf{I} - \mathbf{J})$, we know that $\mathbf{H}(\tilde{\varphi}_j)$ is uniformly bounded from above. So by the compactness Theorem 2.11, $\tilde{\varphi}_j$ converges strongly to $\varphi_\infty \in (\mathcal{E}^1)^{\mathbb{K}}$. By the lower semicontinuity of $\mathbf{M}$ under strong convergence (see [6, Lemma 4.3]), we know that $\varphi_\infty$ is a minimizer of $\mathbf{M}$ over $(\mathcal{E}^1)^{\mathbb{K}}$. Now we can easily adapt [6, Proof of Theorem 4.8] to the $\mathbb{K}$-invariant setting conclude that the $\varphi_\infty$ is a $\mathbb{K}$-invariant Kähler-Einstein metric. □

The last statement of Theorem 2.14 follows from the works of Darvas and Hisamoto via the general framework by Darvas-Rubinstein (in [26]) for proving Tian's properness conjecture from [53]. Note that although Hisamoto's work uses $\mathbf{J}_{C(\mathbb{G})}$ instead of $\mathbf{J}_{C(\mathbb{G})_0}$, the properness conditions using these two norms will turn out to be equivalent. Here we prove a more general result.

**Theorem 2.15.** *Let $(X, D)$ be a log Fano pair. Assume that $\mathrm{Aut}(X)_0$ is reductive, and $\mathbb{G}$ is a connected reductive subgroup of $\mathrm{Aut}(X)_0$ that contains a maximal torus of $\mathrm{Aut}(X)_0$. Then all of the conditions in the above theorem are equivalent.*

*Proof.* We just need to show that condition (3) implies (1). For this, we use Darvas-Rubinstein's principle from [26]. In their notations (see also [24]), we consider the data

$$\mathcal{R} = (\mathcal{E}^1)^{\mathbb{K}} \cap L^\infty(X), \quad \overline{\mathcal{R}} = (\mathcal{E}^1)^{\mathbb{K}}, \quad \mathcal{M} = \{\text{Kähler-Einstein metrics on } (X, D)\},$$

where $\mathbb{K} \subset \mathbb{G}$ is a maximal compact subgroup. We just need to verify that the data $(\mathcal{R}, d_1, \mathbf{D}, \mathbb{T})$ satisfies the properties (P1)-(P7) in [26, Hypothesis 3.2] except for (P5) which needs more argument. The property (P5) means that the space of $\mathbb{K}$-invariant Kähler-Einstein metrics is homogeneous under the action of $\mathbb{T}$ where $\mathbb{T}$ is the identity component of the center of $\mathbb{G}$.

Let $\omega_i, i = 1, 2$ be any two $\mathbb{K}$-invariant Kähler-Einstein metrics and set

$$K_i = \mathrm{Isom}(\omega_i)_0 = \{g \in \mathrm{Aut}_0(X, D); g^*\omega_i = \omega_i\}.$$

Then by [6, section 5], $K_i, i = 1, 2$ are maximal compact subgroups of $\mathrm{Aut}(X, D)_0$. Because $\omega_i$ is $\mathbb{K}$-invariant, we know that $\mathbb{K} \subseteq K_1 \cap K_2$. By assumption, $\mathbb{K}$ contains a maximal compact torus of $\mathbb{G}$. By Proposition A.3, $K_2 = t^{-1}K_1t$ for some $t \in \mathbb{T} = C(\mathbb{G})_0$.

On the other hand, by Berndtsson's theorem (see [6, Appendix C]), there exists $f \in \mathrm{Aut}(X, D)_0 =: \mathfrak{G}$ satisfying $\omega_2 = f^*\omega_1$. So we get $f^{-1}K_1f = K_2 = t^{-1}K_1t$. This implies $ft^{-1} \in N_{\mathfrak{G}}(K_1)$. By Proposition A.1 (see also [40, Proposition 2.13]), $ft^{-1} \in K_1C(\mathfrak{G})_0$. So $f = k_1 \cdot t \cdot t_1 =: k_1 \cdot t'$ for $k_1 \in K_1$, $t \in \mathbb{T}$, $t_1 \in C(\mathfrak{G})_0 \subset \mathbb{T}$ and $t' := t \cdot t_1 \in \mathbb{T}$. So we get $\omega_2 = f^*\omega_1 = t'^*k_1^*\omega_1 = t'^*\omega_1$. We are done. □

## 2.3 Valuations on $T$-varieties

Let $\mathbb{T}$ be a complex torus acting effectively on $Z$. By the structure theory of $\mathbb{T}$-varieties , $Z$ can be described using the language of divisorial fans (see [2, Theorem 5.6]). For us, we just need to know that $Z$ is birationally a torus fibration over the Chow quotient of $Z$ by $\mathbb{T}$ which will be denoted by $Z/\!/\mathbb{T}$. As a consequence the function field $\mathbb{C}(Z)$ is the quotient field of the Laurent polynomial algebra:

$$\mathbb{C}(Z/\!/\mathbb{T})[M_{\mathbb{Z}}] = \bigoplus_{\alpha \in M_{\mathbb{Z}}} \mathbb{C}(Z/\!/\mathbb{T}) \cdot 1^\alpha. \tag{36}$$

Given a valuation $\nu$ of the functional field $\mathbb{C}(Z/\!/\mathbb{T})$ and a vector $\lambda \in N_{\mathbb{R}}$, we obtain a valuation ([2, page 236]):

$$v_{\nu,\lambda} : \mathbb{C}[Z/\!/\mathbb{T}][M_{\mathbb{Z}}] \to \mathbb{R}, \quad \sum_i f_i \cdot 1^{\alpha_i} \quad \mapsto \quad \min \left(\nu(f_i) + \langle \alpha_i, \lambda \rangle\right). \tag{37}$$

In particular, for any $\xi \in N_{\mathbb{R}}$, $\xi$ determines a valuation which will be denoted by $\mathrm{wt}_\xi := v_{\mathrm{triv},\xi}$:

$$\mathrm{wt}_\xi \left( \sum_i f_i \cdot 1^{\alpha_i} \right) = \min_i \langle \alpha_i, \xi \rangle. \tag{38}$$

The vector space $N_{\mathbb{R}}$ acts on $\mathrm{Val}(Z)^{\mathbb{T}}$ in the following natural way. If $v = \nu_{\nu,\lambda}$, then

$$\xi \circ v = \xi \circ v_{\nu,\lambda} = v_{\nu,\lambda+\xi} =: v_\xi. \tag{39}$$

## 2.4 K-stability and Ding-stability

### 2.4.1 Stability via test configurations

In this section we recall the definition of test configurations and stability of log Fano varieties.

**Definition 2.16** ([52, 30], see also [44]). *Let $(Z, Q, L)$ be as before.*

*(1) A test configuration of $(Z, L)$, denoted by $(\mathcal{Z}, \mathcal{L}, \eta)$ or simply by $(\mathcal{Z}, \mathcal{L})$, consists of the following data*

- *A variety $\mathcal{Z}$ admitting a $\mathbb{C}^*$-action which is generated by a holomorphic vector field $\eta$ and a $\mathbb{C}^*$-equivariant morphism $\pi : \mathcal{Z} \to \mathbb{C}$, where the action of $\mathbb{C}^*$ on $\mathbb{C}$ is given by the standard multiplication.*
- *A $\mathbb{C}^*$-equivariant $\pi$-semiample $\mathbb{Q}$-Cartier divisor $\mathcal{L}$ on $\mathcal{Y}$ such that there is an $\mathbb{C}^*$-equivariant isomorphism $\mathfrak{i}_\eta : (\mathcal{Z}, \mathcal{L})|_{\pi^{-1}(\mathbb{C}\setminus\{0\})} \cong (Z, L) \times \mathbb{C}^*$.*

*Let $\mathcal{Q} := Q_{\mathcal{Z}}$ denote the closure of $Q \times \mathbb{C}^*$ in $\mathcal{Z}$ under the inclusion $Q \times \mathbb{C}^* \subset Z \times \mathbb{C}^* \overset{\mathfrak{i}_\eta}{\cong} \mathcal{Z} \times_{\mathbb{C}} \mathbb{C}^* \subset \mathcal{Z}$. We say that $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ is a test configuration of $(Z, Q, L)$.*

*Denote by $\bar{\pi} : (\bar{\mathcal{Z}}, \bar{\mathcal{Q}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$ the natural equivariant compactification of $(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) \to \mathbb{C}$ obtained by using the isomorphism $\mathfrak{i}_\eta$ and then adding a trivial fiber over $\{\infty\} \in \mathbb{P}^1$.*

*(2) A test configuration is called normal if $\mathcal{Z}$ is a normal variety. We will always consider normal test configurations in this paper.*

*A test configuration is called a special test configuration, if the following conditions are satisfied:*

- *$\mathcal{Z}$ is normal, and $\mathcal{Z}_0 = \pi^{-1}(0)$ is an irreducible normal variety;*
- *$\mathcal{L} = -(K_{\mathcal{Z}/\mathbb{C}} + \mathcal{Q})$, which is an $\pi$-ample $\mathbb{Q}$-Cartier divisor;*
- *$(\mathcal{Z}, \mathcal{Z}_0 + \mathcal{Q})$ has plt singularities.*

*A test configuration $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ is called dominating if there exists a $\mathbb{C}^*$-equivariantly birational morphism $\rho : (\mathcal{Z}, \mathcal{Q}) \to (Z, Q) \times \mathbb{C}$.*

*Two test configurations $(\mathcal{Z}_i, \mathcal{Q}_i, \mathcal{L}_i), i = 1, 2$ are called equivalent, if there exists a test configuration $(\mathcal{Z}_3, \mathcal{Q}_3)$ that $\mathbb{C}^*$-equivariantly dominates both test configurations via $q_i : (\mathcal{Z}_3, \mathcal{Q}_3) \to (\mathcal{Z}_i, \mathcal{Q}_i)$, $i = 1, 2$ and satisfies $q_1^*\mathcal{L}_1 = q_2^*\mathcal{L}_2$. Note that any test configuration is equivalent to a dominating test configuration.*

*(3) For any normal test configuration $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ of $(Z, Q, L)$, define the divisor $\Delta_{(\mathcal{Z},\mathcal{Q},\mathcal{L})}$ to be the $\mathbb{Q}$-divisor supported on $\mathcal{Z}_0$ that is given by:*

$$\Delta := \Delta_{(\mathcal{Z},\mathcal{Q},\mathcal{L})} = -K_{\mathcal{Z}/\mathbb{C}} - \mathcal{Q} - \mathcal{L}. \tag{40}$$

Set $V = L^{\cdot n}$ to be the volume. For any (dominating) normal test configuration $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$, we attach the following well-known invariants:

$$\mathbf{E}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}) = \frac{1}{V}\frac{\left(\bar{\mathcal{L}}^{\cdot n+1}\right)}{n+1}, \tag{41}$$

$$\mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}) = \frac{1}{V}\left(\bar{\mathcal{L}} \cdot \rho^*(L \times \mathbb{P}^1)^{\cdot n}\right), \tag{42}$$

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}) = \frac{1}{V}\left(\bar{\mathcal{L}} \cdot \rho^*(L \times \mathbb{P}^1)^{\cdot n}\right) - \frac{1}{V}\frac{\left(\bar{\mathcal{L}}^{\cdot n+1}\right)}{n+1}, \tag{43}$$

$$\mathrm{CM}(\mathcal{Z}, \mathcal{L}) := \mathrm{CM}(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) = \frac{1}{(n+1)V}\left(n\bar{\mathcal{L}}^{\cdot n+1} + (n+1)\bar{\mathcal{L}}^{\cdot n} \cdot K_{(\bar{\mathcal{Z}},\bar{\mathcal{Q}})/\mathbb{P}^1}\right), \tag{44}$$

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}) := \mathbf{L}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) = \mathrm{lct}(\mathcal{Z}, \mathcal{Q} + \Delta; \mathcal{Z}_0) - 1, \tag{45}$$

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}) := \mathbf{D}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) = \frac{-\bar{\mathcal{L}}^{\cdot n+1}}{(n+1)V} + \left(\mathrm{lct}(\mathcal{Z}, \mathcal{Q} + \Delta; \mathcal{Z}_0) - 1\right). \tag{46}$$

**Remark 2.17.** *There is an explicit and useful formula for* $\mathbf{L}^{\mathrm{NA}}(\mathcal{Z},\mathcal{Q},\mathcal{L})$*. Choose a* $\mathbb{C}^*$*-equivariant log resolution* $\pi_{\mathcal{Z}}:\mathcal{U}\to(\mathcal{Z},\mathcal{Q})$ *such that* $(\mathcal{Z},\mathcal{Z}_0+\pi_{\mathcal{Z}}^{-1}(\mathcal{Q}))$ *is a log smooth pair. Write:*

$$K_{\mathcal{U}}=\pi_{\mathcal{Z}}^*(K_{\mathcal{Z}}+\mathcal{Q})+\sum_i a_iE_i+\sum_j a'_jE'_j,\quad \pi^*\mathcal{Z}_0=\sum_i b_iE_i,\quad \pi^*\Delta=\sum_i c_iE_i,$$

*where* $E_i$ *are vertical divisors and* $E'_j$ *are horizontal divisors. Then we have the following formula (see [4, Proposition 3.8]):*

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{Z},\mathcal{Q},\mathcal{L})=\min_i\frac{a_i-c_i+1}{b_i}-1. \tag{47}$$

*In particular, this means that* $\mathrm{lct}(\mathcal{Z},\mathcal{Q}+\Delta;\mathcal{Z}_0)$ *is calculated by some* $E_i$ *whose center over* $\mathcal{Z}$ *is supported on* $\mathcal{Z}_0$*.*

The following result is now well known:

**Proposition 2.18** (see [15])**.** *Let* $(\mathcal{Z},\mathcal{Q},\mathcal{L})$ *be a normal test configuration of* $(Z,Q,L)$*. Let* $\Phi=\{\varphi(t)\}$ *be a bounded and positively curved Hermitian metric on* $\mathcal{L}$*. Then the following limits hold true:*

$$\mathbf{F}'^{\infty}(\Phi):=\lim_{t\to0}\frac{\mathbf{F}(\varphi(t))}{-\log|t|^2}=\mathbf{F}^{\mathrm{NA}}(\mathcal{Z},\mathcal{Q},\mathcal{L}), \tag{48}$$

*where the energy* $\mathbf{F}$ *is any one from* $\{\mathbf{E},\mathbf{\Lambda},\mathbf{J},\mathbf{L},\mathbf{D}\}$*.*

**Definition 2.19.** *(1)* $(Z,Q)$ *is called uniformly K-stable if there exists* $\gamma>0$ *such that* $\mathrm{CM}(\mathcal{Z},\mathcal{Q},\mathcal{L})\ge\gamma\cdot\mathbf{J}^{\mathrm{NA}}(\mathcal{Z},\mathcal{L})$ *for any normal test configuration* $(\mathcal{Z},\mathcal{Q},\mathcal{L})/\mathbb{C}$ *of* $(Z,Q,L)$*.*

*(2)* $(Z,Q)$ *is called uniformly Ding-stable if there exists* $\gamma>0$ *such that* $\mathbf{D}^{\mathrm{NA}}(\mathcal{Z},\mathcal{Q},\mathcal{L})\ge\gamma\cdot\mathbf{J}^{\mathrm{NA}}(\mathcal{Z},\mathcal{L})$ *for any normal test configuration* $(\mathcal{Z},\mathcal{Q},\mathcal{L})/\mathbb{C}$ *of* $(Z,Q,L)$*.*

*For convenience, we will call* $\gamma$ *to be a slope constant.*

For any special test configuration $(\mathcal{Z}^s,\mathcal{Q}^s,\mathcal{L}^s)$, its CM weight coincides with its $\mathbf{D}^{\mathrm{NA}}$ invariant, which coincides with the original Futaki invariant (as generalized by Ding-Tian):

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{Z}^s,\mathcal{Q}^s,\mathcal{L}^s)=\mathrm{CM}(\mathcal{Z}^s,\mathcal{Q}^s,\mathcal{L}^s)=-\frac{(-K_{(\overline{\mathcal{Z}^s},\overline{\mathcal{Q}^s})/\mathbb{P}^1})^{\cdot n+1}}{(n+1)L^{\cdot n}}=\mathrm{Fut}_{(\mathcal{Z}^s_0,\mathcal{Q}^s_0)}(\eta). \tag{49}$$

By the work in [7, 33] (see also [44]), to test uniform K-stability, one only needs to test on special test configurations. As a consequence,

**Theorem 2.20** ([7, 33])**.** *For a log Fano pair* $(X,D)$*,* $(X,D)$ *is uniformly K-stable if and only if* $(X,D)$ *is uniformly Ding-stable.*

#### 2.4.2 Stability via filtrations

We here briefly recall the relevant definitions about filtrations and refer the details to [12] (see also [14]). For any integer $\ell_0$ such that $-\ell_0(K_Z+Q)=\ell_0L$ is Cartier, we set:

$$R_m^{(\ell_0)}:=H^0(X,m\ell_0L),\quad R^{(\ell_0)}:=\bigoplus_{m=0}^{+\infty}R_m^{(\ell_0)},\quad N_m^{(\ell_0)}:=\dim_{\mathbb{C}}R_m^{(\ell_0)}. \tag{50}$$

If the integer $\ell_0$ is clear, we also denote the above data by $R_m,R,N_m$.

**Definition 2.21.** *A filtration* $\mathcal{F}R_\bullet$ *of the graded* $\mathbb{C}$*-algebra* $R=\bigoplus_{m=0}^{+\infty}R_m$ *consists of a family of subspaces* $\{\mathcal{F}^xR_m\}_x$ *of* $R_m$ *for each* $m\ge0$ *satisfying:*

- *(decreasing)* $\mathcal{F}^xR_m\subseteq\mathcal{F}^{x'}R_m$*, if* $x\ge x'$*;*

- *(left-continuous)* $\mathcal{F}^x R_m = \bigcap_{x'<x} \mathcal{F}^{x'} R_m$;
- *(multiplicative)* $\mathcal{F}^x R_m \cdot \mathcal{F}^{x'} R_{m'} \subseteq \mathcal{F}^{x+x'} R_{m+m'}$, *for any* $x, x' \in \mathbb{R}$ *and* $m, m' \in \mathbb{Z}_{\geq 0}$;
- *(linearly bounded) There exist* $e_-, e_+ \in \mathbb{Z}$ *such that* $\mathcal{F}^{me_-} R_m = R_m$ *and* $\mathcal{F}^{me_+} R_m = 0$ *for all* $m \in \mathbb{Z}_{\geq 0}$.

*We say that* $\mathcal{F}$ *is a* $\mathbb{Z}$*-filtration if* $\mathcal{F}^x R_m = \mathcal{F}^{\lceil x \rceil} R_m$ *for each* $x \in \mathbb{R}$ *and* $m \in \mathbb{Z}_{\geq 0}$.

*Given such a filtration* $\mathcal{F}$, *for any* $\theta \in \mathbb{R}$, *the* $\theta$*-shifting of* $\mathcal{F}$, *denoted by* $\mathcal{F}(\theta)$ *is defined to be the filtration given by:*

$$\mathcal{F}(\theta)^x R_m := \mathcal{F}^{x - m\ell_0\theta} R_m. \tag{51}$$

Given any filtration $\{\mathcal{F}^x R_m\}_{x\in\mathbb{R}}$ and $m \in \mathbb{Z}_{\geq 0}$, the successive minima on $R_m$ is the decreasing sequence

$$\lambda^{(m)}_{\max} = \lambda^{(m)}_1 \geq \cdots \geq \lambda^{(m)}_{N_m} = \lambda^{(m)}_{\min}$$

defined by:

$$\lambda^{(m)}_j = \max\left\{\lambda \in \mathbb{R}; \dim_{\mathbb{C}} \mathcal{F}^\lambda R_m \geq j\right\}.$$

If $\{\mathcal{F}^x R_m\}_x$ is a $\mathbb{Z}$-filtration, then $\{\mathcal{F}^x R_m\}_x$ can be equivalently described as a $\mathbb{C}^*$-equivariant degeneration of $R_m$. More precisely, there is a $\mathbb{C}^*$-equivariant vector bundle $\mathcal{R}_m$ over $\mathbb{C}$ such that

$$\mathcal{R}_m \times_{\mathbb{C}} \mathbb{C}^* \cong R_m \times \mathbb{C}^*, \quad (\mathcal{R}_m)_0 = \bigoplus_{i=0}^{+\infty} \mathcal{F}^{\lambda^{(m)}_{i+1}} R_m / \mathcal{F}^{\lambda^{(m)}_i} R_m. \tag{52}$$

Denote $\mathcal{F}^{(t)} := \mathcal{F}^{(t)} R = \bigoplus_{k=0}^{+\infty} \mathcal{F}^{kt} R_k$ and define

$$\mathrm{vol}\left(\mathcal{F}^{(t)}\right) = \mathrm{vol}\left(\mathcal{F}^{(t)} R\right) := \limsup_{k\to+\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{mt} H^0(Z, m\ell_0 L)}{m^n/n!}. \tag{53}$$

The following results are very useful.

**Proposition 2.22** ([12], [14, Corollary 5.4])**.** *(1) The probability measure*

$$\frac{1}{N_m} \sum_j \delta_{m^{-1}\lambda^{(m)}_j} = -\frac{d}{dt} \frac{\dim_{\mathbb{C}} \mathcal{F}^{mt} H^0(Z, m\ell_0 L)}{N_m}$$

*converges weakly as* $m \to +\infty$ *to the probability measure:*

$$\mathrm{DH}(\mathcal{F}) := -\frac{1}{\ell_0^n L^{\cdot n}} d\,\mathrm{vol}\left(\mathcal{F}^{(t)}\right) = -\frac{1}{\ell_0^n L^{\cdot n}} \frac{d}{dt} \mathrm{vol}\left(\mathcal{F}^{(t)}\right) dt.$$

*(2) The support of the measure* $\mathrm{DH}(\mathcal{F})$ *is given by* $\mathrm{supp}(\mathrm{DH}(\mathcal{F})) = [\lambda^{(\ell_0)}_{\min}, \lambda^{(\ell_0)}_{\max}]$ *with*

$$\lambda_{\min} := \lambda_{\min}(\mathcal{F}) := \inf\left\{t \in \mathbb{R}; \mathrm{vol}\left(\mathcal{F}^{(t)}\right) < \ell_0^n L^{\cdot n}\right\}; \tag{54}$$

$$\lambda_{\max} := \lambda_{\max}(\mathcal{F}) := \lim_{m\to+\infty} \frac{\lambda^{(m)}_{\max}}{m} = \sup_{m\geq 1} \frac{\lambda^{(m)}_{\max}}{m}. \tag{55}$$

For a filtration $\mathcal{F}R_\bullet$, choose $e_-$ and $e_+$ as in the definition 2.21. For convenience, we can choose $e_+ = \lceil \lambda_{\max}(\mathcal{F}R) \rceil \in \mathbb{Z}$. Set $e = e_+ - e_-$ and define (fractional) ideals:

$$I_{m,x} := I^{\mathcal{F}}_{m,x} := \mathrm{Image}\left(\mathcal{F}^x R_m \otimes \mathcal{O}_Z(m\ell_0 L) \to \mathcal{O}_Z\right); \tag{56}$$

$$\tilde{\mathcal{I}}_m := \tilde{\mathcal{I}}^{\mathcal{F}}_m := I^{\mathcal{F}}_{(m,me_+)} t^{-me_+} + I^{\mathcal{F}}_{(m,me_+-1)} t^{1-me_+} + \cdots + I^{\mathcal{F}}_{(m,me_-+1)} t^{-me_--1} + \mathcal{O}_Z \cdot t^{-me_-}; \tag{57}$$

$$\mathcal{I}_m := \mathcal{I}^{\mathcal{F}(e_+)}_m = \tilde{\mathcal{I}}^{\mathcal{F}}_m \cdot t^{me_+} = I^{\mathcal{F}}_{(m,me_+)} + I^{\mathcal{F}}_{(m,me_+-1)} t^1 + \cdots + I^{\mathcal{F}}_{(m,me_-+1)} t^{me-1} + (t^{me}) \subseteq \mathcal{O}_{Z_{\mathbb{C}}}. \tag{58}$$

**Definition-Proposition 2.23** ([32, Lemma 4.6]). *With the above notations, for $m$ sufficiently divisible, define the $m$-th approximating test configuration $(\check{\mathcal{Z}}^{\mathcal{F}}_m, \check{\mathcal{Q}}^{\mathcal{F}}_m, \check{\mathcal{L}}^{\mathcal{F}}_m)$ as:*

*(1) $\check{\mathcal{Z}}^{\mathcal{F}}_m$ is the normalization of blowup of $Z\times\mathbb{C}$ along the ideal sheaf $\mathcal{I}_m^{\mathcal{F}(e_+)}$;*

*(2) $\check{\mathcal{Q}}^{\mathcal{F}}_m$ is the closure of $Q\times\mathbb{C}^*$ under the $\mathbb{C}^*$-equivariant inclusion $Q\times\mathbb{C}^*\subset Z\times\mathbb{C}^*\subset\mathcal{Z}$;*

*(3) The semiample $\mathbb{Q}$-divisor is given by:*

$$\check{\mathcal{L}}^{\mathcal{F}}_m=\pi^*(L\times\mathbb{C})-\frac{1}{m\ell_0}E_m+\frac{e_+}{\ell_0}\check{\mathcal{Z}}_0, \tag{59}$$

*where $E_m$ is the exceptional divisor of the normalized blow up.*

*For simplicity of notations, we also denote the data by $(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m)$ if the filtration is clear. Note that $m\ell_0\check{\mathcal{L}}_m$ is Cartier over $\check{\mathcal{Z}}_m$.*

We will be interested in the following invariants attached to filtrations:

$$\mathbf{E}^{\rm NA}(\mathcal{F}) = \int_{\lambda_{\min}}^{+\infty}\frac{x}{\ell_0}\cdot {\rm DH}(\mathcal{F})=\lim_{m\to+\infty}\frac{1}{N_m}\sum_{j=1}^{N_m}\frac{\lambda_j^{(m)}}{m\ell_0}; \tag{60}$$

$$\mathbf{\Lambda}^{\rm NA}(\mathcal{F}) = \lim_{m\to+\infty}\frac{\lambda^{(m)}_{\max}(\mathcal{F})}{m\ell_0}=\sup_{m\ge1}\frac{\lambda^{(m)}_{\max}(\mathcal{F})}{m\ell_0}; \tag{61}$$

$$\mathbf{J}^{\rm NA}(\mathcal{F}) = \mathbf{\Lambda}^{\rm NA}(\mathcal{F})-\mathbf{E}^{\rm NA}(\mathcal{F}); \tag{62}$$

$$\mathbf{L}^{\rm NA}(\mathcal{F}) := {\rm lct}\left(Z\times\mathbb{C}, Q\cdot\left(\mathcal{I}_\bullet^{\mathcal{F}(e_+)}\right)^{\frac{1}{\ell_0}};(t)\right)+\frac{e_+}{\ell_0}-1; \tag{63}$$

$$\mathbf{D}^{\rm NA}(\mathcal{F}) := -\mathbf{E}^{\rm NA}(\mathcal{F})+\mathbf{L}^{\rm NA}(\mathcal{F}). \tag{64}$$

In the above definition of $\mathbf{L}^{\rm NA}$, we used the following notations (see [41] for the definition of log canonical thresholds of graded sequence of ideals):

$$\begin{aligned}
&{\rm lct}\left(Z\times\mathbb{C},Q\cdot\left(\mathcal{I}_\bullet^{\mathcal{F}(e_+)}\right)^{\frac{1}{\ell_0}};(t)\right)=\lim_{m\to+\infty}{\rm lct}\left(\left(Z\times\mathbb{C},Q\cdot\mathcal{I}_m^{\mathcal{F}(e_+)}\right)^{\frac{1}{m\ell_0}};(t)\right);\\
&{\rm lct}\left(Z\times\mathbb{C},Q\cdot\left(\mathcal{I}_m^{\mathcal{F}(e_+)}\right)^{\frac{1}{m\ell_0}};(t)\right)\\
&\qquad=\sup\left\{c;\left(Z\times\mathbb{C},Q\cdot\left(\mathcal{I}_m^{\mathcal{F}(e_+)}\right)^{\frac{1}{m\ell_0}}\cdot(t)^c\right)\text{ is sub-log-canonical }\right\}.
\end{aligned}$$

**Example 2.24.** *Assume $(\mathcal{Z},\mathcal{Q},\mathcal{L})$ is a test configuration of $(Z,Q,L)$. Choose $\ell_0>0$ such that $\ell_0\mathcal{L}$ is Cartier. Then we have an associated $\mathbb{Z}$-filtration $\mathcal{F}=\mathcal{F}_{(\mathcal{Z},\ell_0\mathcal{L})}$ on $R=R^{(\ell_0)}$ defined in the following way:*

*$s\in\mathcal{F}^xR^{(\ell_0)}_m$ if and only if $t^{-\lceil x\rceil}\bar{s}$ extends to a holomorphic section of $m\ell_0\mathcal{L}$, where $\bar{s}$ is the meromorphic section of $m\ell_0\mathcal{L}$ defined as the pull-back of $s$ via the projection $(\mathcal{Z},\mathcal{L})\times_{\mathbb{C}}\mathbb{C}^*\cong(Z,L)\times\mathbb{C}^*\to Z$. Assume the test configuration is dominating and write $\mathcal{L}=\rho^*L_{\mathbb{C}}+D$ (see Definition 2.16) where $L_{\mathbb{C}}=p_1^*L$. Then by [14, Lemma 5.17], this filtration has the following more explicit description:*

$$\mathcal{F}^xR_m=\bigcap_E\{s\in H^0(Z,m\ell_0L); r({\rm ord}_E)(s)+m\ell_0\,{\rm ord}_E(D)\ge xb_E\}, \tag{65}$$

*where $E$ runs over the irreducible components of the central fibre $\mathcal{Z}_0$, $b_E={\rm ord}_E(\mathcal{Z}_0)={\rm ord}_E(t)$ and $r({\rm ord}_E)$ denotes the restriction of ${\rm ord}_E$ to $\mathbb{C}(Z)$ under the inclusion $\mathbb{C}(Z)\subset\mathbb{C}(X\times\mathbb{C}^*)=\mathbb{C}(\mathcal{X})$.*

*For this filtration, we have* $\mathbf{F}^{\rm NA}(\mathcal{F}) = \mathbf{F}^{\rm NA}(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ *for* $\mathbf{F}$ *being the functionals defined in* (60)-(64)*. For* $m$ *sufficiently divisible we have (see [14, Theorem 5.18 and Lemma 7.7])*

$$\mathbf{\Lambda}^{\rm NA}(\mathcal{Z}, \mathcal{L}) = \frac{\lambda_{\max}(\mathcal{F}_{(\mathcal{Z},\ell_0\mathcal{L})})}{\ell_0} = \frac{\lambda^{(m)}_{\max}(\mathcal{F}_{(\mathcal{Z},\ell_0\mathcal{L})})}{\ell_0 m} = \frac{1}{V}\rho^*(L\times\mathbb{P}^1)^{\cdot n}\cdot\overline{\mathcal{L}}. \tag{66}$$

*Moreover, because* $\mathcal{F}_{(\mathcal{Z},\ell_0\mathcal{L})}$ *is finitely generated (see [55, 51, 14]), for* $m$ *sufficiently divisible, the* $m$*-th approximating test configurations* $(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m)$ *are equivalent to* $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$*.*

**Example 2.25.** *Given any valuation* $v \in \mathring{\rm Val}(Z)$*, we have an associated filtration* $\mathcal{F} = \mathcal{F}_v$*:*

$$\mathcal{F}^x_v R_m := \{s \in R_m; v(s) \geq x\}. \tag{67}$$

*The following quantity plays an important role in recent studies of K-stability (see e.g. [33, 43, 11]):*

$$S_L(v) = \frac{1}{\ell_0^{n+1}L^{\cdot n}}\int_0^{+\infty}{\rm vol}(\mathcal{F}^{(x)}_v R)dx =: \frac{1}{L^{\cdot n}}\int_0^{+\infty}{\rm vol}(L-tv)dt, \tag{68}$$

*where we have denoted by* ${\rm vol}(L-tv)$ *the quantity* ${\rm vol}(\mathcal{F}^{(t\ell_0)}_v R^{(\ell_0)})/\ell_0^{n+1}$*.*

*By Izumi's inequality (see [41, 42]), there exist* $c_1, c_2 > 0$ *such that* $c_1\cdot{\rm ord}_W \leq v \leq c_2 A_{(Z,Q)}(v){\rm ord}_W$ *where* $W$ *is the center of* $v$*. So we get* $\lambda_{\min}(\mathcal{F}_v) = 0$*. Then by integration by parts we get:*

$$\mathbf{E}^{\rm NA}(\mathcal{F}_v) = -\frac{1}{\ell_0^n L^{\cdot n}}\int_0^{+\infty}\frac{x}{\ell_0}\cdot d{\rm vol}(\mathcal{F}^{(x)}R) = S_L(v). \tag{69}$$

*Moreover, by [34, Proposition 2.1] (see also [17, (5.3)]), we have a very useful inequality:*

$$\frac{1}{n}S_L(v) \leq \mathbf{J}^{\rm NA}(\mathcal{F}_v) = \mathbf{\Lambda}^{\rm NA}(\mathcal{F}_v) - S_L(v) \leq nS_L(v). \tag{70}$$

**Example 2.26.** *Assume that a complex torus* $\mathbb{T}$ *acts on* $(Z, L)$*. Then we have a weight decomposition:*

$$R_m = \bigoplus_{\alpha\in M_{\mathbb{Z}}}(R_m)_\alpha = (R_m)_{\alpha^{(m)}_1}\oplus\cdots\oplus(R_m)_{\alpha^{(m)}_{N_m}}. \tag{71}$$

*For any* $\xi \in N_{\mathbb{R}}$*, let* $\kappa^{(m)}_j = \langle\alpha^{(m)}_j, \xi\rangle, j = 1, \ldots, N_m$ *be the weight of* $\xi$ *on* $R_m$*. The Chow weight of* $\xi$ *on* $L$ *is then defined as:*

$${\rm CW}_L(\xi) := \lim_{m\to+\infty}\frac{1}{N_m}\sum_j\frac{\kappa^{(m)}_j}{m\ell_0}. \tag{72}$$

*In our set-up we have* $L = -K_Z - Q$ *with the canonical* $\mathbb{T}$*-action, then (e.g. from* (49)*)*

$${\rm CW}_L(\xi) = -{\rm Fut}_{(Z,Q)}(\xi). \tag{73}$$

*On the other hand,* $\xi$ *determines a valuation* ${\rm wt}_\xi$*. Now let* $W$ *be the center of* ${\rm wt}_\xi$ *and* $U$ *be a* $\mathbb{T}$*-invariant Zariski open set such that* $U\cap W \neq \emptyset$*. Let* $\mathfrak{e}$ *be an* $\mathbb{T}$*-equivariant non-vanishing generator of* $\mathcal{O}_Z(\ell_0 L)$ *and let* $\mathbf{w} = \frac{\mathscr{L}_\xi\mathfrak{e}}{\mathfrak{e}}$*. Then we have:*

$$\mathbf{E}^{\rm NA}(\mathcal{F}_{{\rm wt}_\xi}) = \frac{1}{N_m}\lim_{m\to+\infty}\sum_j\frac{\kappa^{(m)}_j}{m\ell_0} - \frac{\mathscr{L}_\xi\mathfrak{e}}{\mathfrak{e}} = {\rm CW}_L(\xi) - \mathbf{w}. \tag{74}$$

**Lemma 2.27** (see [14, Lemma 5.17])**.** *The filtrations in the above examples are saturated. In other words, for* $m$ *sufficiently divisible, we have:*

$$\mathcal{F}^x R^{(\ell_0)}_m = H^0\left(Z, \mathcal{O}_Z(-mK_Z\otimes I^{\mathcal{F}}_{m,x})\right). \tag{75}$$

To characterize Ding stability via filtrations, the following lemma is crucial.

**Proposition 2.28** ([33, Lemma 4.3], [32, Lemma 4.7], [16, Theorem 4.13])**.** *For any valuation $v \in \mathring{\mathrm{Val}}(X)$ and $\mathcal{F} = \mathcal{F}_v$, let $(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m)$ be the same as in Definition 2.23. Then the following limits hold true:*

$$\lim_{m\to+\infty} \mathbf{\Lambda}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m) = \mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{F}); \tag{76}$$

$$\lim_{m\to+\infty} \mathbf{E}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m) = \mathbf{E}^{\mathrm{NA}}(\mathcal{F}); \tag{77}$$

$$\lim_{m\to+\infty} \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m) = \mathbf{J}^{\mathrm{NA}}(\mathcal{F}). \tag{78}$$

We state and sketch a proof of a result of Fujita, which will be generalized to the equivariant case.

**Theorem 2.29** ([32])**.** *Assume that $(Z, Q)$ is uniformly Ding-stable. Then there exists $\gamma > 0$ such that for any filtration $\mathcal{F} = \mathcal{F}_v$,*

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{F}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{F}). \tag{79}$$

*Proof.* By construction, we have the identity:

$$\mathbf{L}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m) = \mathrm{lct}\left(Z \times \mathbb{C}, Q \cdot (\mathcal{I}^{\mathcal{F}}_m)^{\frac{1}{\ell_0 m}}; (t)\right) - 1. \tag{80}$$

As a consequence,

$$\lim_{m\to+\infty} \mathbf{L}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m) = \mathrm{lct}\left(Z \times \mathbb{C}, Q \cdot (\mathcal{I}^{\mathcal{F}}_\bullet)^{\frac{1}{\ell_0}}; (t)\right) - 1 = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}).$$

Combining this with (77) and using $\mathbf{D}^{\mathrm{NA}} = -\mathbf{E}^{\mathrm{NA}} + \mathbf{L}^{\mathrm{NA}}$, we get the limit:

$$\lim_{m\to+\infty} \mathbf{D}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m) = \mathbf{D}^{\mathrm{NA}}(\mathcal{F}). \tag{81}$$

If $Z$ is uniformly Ding-stable with a slope constant $\gamma$, then $\mathbf{D}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m) \geq \gamma \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m)$. The conclusion follows by letting $m \to +\infty$ and using Lemma 2.28. □

#### 2.4.3 Boucksom-Jonsson's non-Archimedean formulation

Here we briefly recall the non-Archimedean formulation after Boucksom-Jonsson. Let $(Z, Q, L)$ be the polarized projective variety as before. We denote by $(Z^{\mathrm{NA}}, Q^{\mathrm{NA}}, L^{\mathrm{NA}})$ the Berkovich analytification of $(Z, Q, L)$ with respect to the trivial absolute value on the ground field $\mathbb{C}$. $Z^{\mathrm{NA}}$ is a topological space, whose points can be considered as semivaluations on $Z$, i.e. valuations $v : \mathbb{C}(W)^* \to \mathbb{R}$ on function field of subvarieties $W$ of $Z$, trivial on $\mathbb{C}$. In particular, $\mathring{\mathrm{Val}}(Z) \subset Z^{\mathrm{NA}}$. The topology of $Z^{\mathrm{NA}}$ is generated by functions of the form $v \mapsto v(f)$ with $f$ a regular function on some Zariski open set $U \subset Z$. One can show that $Z^{\mathrm{NA}}$ is compact and Hausdorff, and $\mathring{\mathrm{Val}}(Z) \subset Z^{\mathrm{NA}}$ is dense.

In this paper, we will only use non-Archimedean metrics on $L^{\mathrm{NA}}$ coming from test configurations and filtrations. Moreover we will always identify a non-Archimedean metrics with functions on $\mathring{\mathrm{Val}}(Z)$.

For any $w \in \mathring{\mathrm{Val}}(Z)$, let $G(w)$ denote the standard Gauss extension: for any $f = \sum_{i\in\mathbb{Z}} f_i t^i \in \mathbb{C}(Z \times \mathbb{C})$ with $f_i \in \mathbb{C}(Z)$,

$$G(w)\left(\sum_i f_i t^i\right) = \min_i \{w(f_i) + i\} \tag{82}$$

**Definition 2.30.** *Let $(\mathcal{Z}, \mathcal{L})$ be a dominating test configuration of $(Z, L)$ with $\rho : \mathcal{Z} \to Z \times \mathbb{C}$ being a $\mathbb{C}^*$-equivariant morphism. The non-Archimedean metric defined by $(\mathcal{Z}, \mathcal{L})$ is given by the following function on $\mathring{\mathrm{Val}}(Z)$:*

$$\phi_{(\mathcal{Z},\mathcal{L})}(w) = G(w) \left(\mathcal{L} - \rho^*(L \times \mathbb{C})\right). \tag{83}$$

*If $(\mathcal{Z}, \mathcal{L})$ is obtained as blowups of $(Z, L) \times \mathbb{C}$ along some flag ideal sheaf $\mathcal{I}$:*

$$\mathcal{Z} = \text{ normalization of } Bl_{\mathcal{I}}(Z \times \mathbb{C}), \quad \mathcal{L} = \pi^* L \times \mathbb{C} - cE \tag{84}$$

*for some $c \in \mathbb{Q} > 0$, where $\pi : \mathcal{Z} \to Z \times \mathbb{C}$ is the natural projection and $E$ is the exceptional divisor of blowup, then we have:*

$$\phi_{(\mathcal{Z},\mathcal{L})}(w) = -G(w)(cE) = -c \cdot G(w)(\mathcal{I}). \tag{85}$$

*The set of non-Archimedean metrics obtained in such a way will be denoted as $\mathcal{H}^{\mathrm{NA}}(L)$.*

**Definition 2.31.** *Let $\mathcal{F} = \mathcal{F}R_\bullet$ be a filtration. For any $w \in \mathring{\mathrm{Val}}(Z)$, define the non-Archimedean metric associated to $\mathcal{F}$ as:*

$$\begin{aligned} \phi^{\mathcal{F}}_m(w) &= -\frac{1}{m} G(w) \left( \left(\tilde{\mathcal{I}}^{\mathcal{F}}_m\right)^{\frac{1}{\ell_0}} \right) = -\frac{1}{m} G(w) \left( \left(\mathcal{I}^{\mathcal{F}(e_+)}_m t^{-me_+}\right)^{\frac{1}{\ell_0}} \right) \\ &= -\frac{1}{\ell_0}\frac{1}{m} G(w) \left(\mathcal{I}^{\mathcal{F}(e_+)}_m\right) + \frac{e_+}{\ell_0}; \end{aligned} \tag{86}$$

$$\phi^{\mathcal{F}}(w) = -G(w) \left( \left(\tilde{\mathcal{I}}^{\mathcal{F}}_\bullet\right)^{\frac{1}{\ell_0}} \right) = \lim_{m \to +\infty} \phi^{\mathcal{F}}_m(w). \tag{87}$$

*In particular, if $v \in \mathring{\mathrm{Val}}(Z)$ and $\mathcal{F} = \mathcal{F}_v$, then we denote $\phi_v = \phi^{\mathcal{F}_v}$.*

Note that from the definition 2.31 and 2.23 we see that:

$$\phi^{\mathcal{F}}_m = \phi_{(\check{\mathcal{Z}}^{\mathcal{F}}_m, \check{\mathcal{L}}^{\mathcal{F}}_m)}. \tag{88}$$

**Lemma 2.32** (see [17, Theorem 5.13])**.** *For any $v \in \mathring{\mathrm{Val}}(Z)$, $\phi_v$ satisfies $\phi_v(v) = 0$ and $(\omega^{\mathrm{NA}}_{\phi_v})^n = \delta_v$.*

In this paper, we only need the fact that $\phi_v(v) = 0$ which can be verified directly from the definition. The non-Archimedean functionals are defined formally as:

$$\mathbf{E}^{\mathrm{NA}}(\phi) := \mathbf{E}^{\mathrm{NA}}_L(\phi) = \frac{1}{(n+1)(2\pi)^n L^{\cdot n}} \sum_{j=0}^{n} \int_{Z^{\mathrm{NA}}} \phi (\omega^{\mathrm{NA}}_\phi)^j \wedge (\omega^{\mathrm{NA}})^{n-j}, \tag{89}$$

$$\mathbf{J}^{\mathrm{NA}}(\phi) := \mathbf{J}^{\mathrm{NA}}_L(\phi) = \frac{1}{(2\pi)^n L^{\cdot n}} \int_{Z^{\mathrm{NA}}} \phi \cdot (\omega^{\mathrm{NA}})^n - \mathbf{E}^{\mathrm{NA}}(\phi), \tag{90}$$

$$\mathbf{L}^{\mathrm{NA}}(\phi) := \mathbf{L}^{\mathrm{NA}}_{(Z,Q)}(\phi) = \inf_{w \in Z^{\mathrm{div}}_{\mathbb{Q}}} \left(A_{(Z,Q)}(w) + \phi(w)\right). \tag{91}$$

They recover the non-Archimedean functional for test configurations and for filtrations: for functional $\mathbf{F}$ appearing in (41)-(46) and (60)-(64):

$$\mathbf{F}^{\mathrm{NA}}(\phi_{(\mathcal{Z},\mathcal{L})}) = \mathbf{F}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}), \quad \mathbf{F}^{\mathrm{NA}}(\phi^{\mathcal{F}}) = \mathbf{F}^{\mathrm{NA}}(\mathcal{F}). \tag{92}$$

Later we will also use the fact that the multiplicative group $\mathbb{R}^\times_+$ acts on the space of non-Archimedean metrics that come from filtrations. For any $b > 0$ and a non-Archimedean metric that is represented by a function $\phi$ on $\mathring{\mathrm{Val}}(Z)$, the action is given by the formula (see [16, (2.1)]):

$$(b \circ \phi)(v) = b \cdot \phi(b^{-1} v). \tag{93}$$

In the case that $\phi = \phi_{(\mathcal{Z},\mathcal{L})}$ and $b \in \mathbb{Z}_{>0}$, the rescaling operation corresponds to the base change. To see this we denote

$$(\mathcal{Z},\mathcal{Q},\mathcal{L})^{(b)} := (\text{ normalization of } (\mathcal{Z},\mathcal{Q},\mathcal{L}) \times_{\mathbb{C},\mathrm{m}_b} \mathbb{C}, b\cdot\eta) \xrightarrow{\pi_b} (\mathcal{Z},\mathcal{Q},\mathcal{L}), \tag{94}$$

where $\mathrm{m}_b : t' \to t'^b = t$, $b \cdot \eta := b \cdot \mathrm{m}_b^*\eta$. Then it is easy to verify that $(\pi_b)_* G(v) = bG(b^{-1}v)$ so that

$$\begin{aligned}\phi_{(\mathcal{Z},\mathcal{L})^{(b)}}(v) &= G(v)(\pi_b^*(\mathcal{L} - \rho^*(L\times\mathbb{C}))) = (\pi_b)_*G(v)(\mathcal{L} - \rho^*(L\times\mathbb{C})) \\ &= bG(b^{-1}v)(\mathcal{L} - \rho^*(L\times\mathbb{C})) = b\phi_{(\mathcal{Z},\mathcal{L})}(b^{-1}v) = (b\circ\phi_{(\mathcal{Z},\mathcal{L})})(v).\end{aligned} \tag{95}$$

# 3 Twists of non-Archimedean metrics

## 3.1 Twists of test configurations

Let $(Z, Q)$ be as before. Assume $\mathbb{G}$ is a reductive complex Lie group that acts faithfully on $(Z, Q)$. Then $\mathbb{G}$ naturally acts on $L := -K_Z - Q$.

**Definition 3.1.** *$(\mathcal{Z},\mathcal{Q},\mathcal{L},\eta)$ is a $\mathbb{G}$-equivariant test configuration of $(Z, Q, L)$ if*

- *$(\mathcal{Z},\mathcal{Q},\mathcal{L})$ is a test configuration of $(Z, Q, L)$;*
- *$\mathbb{G}$ acts on $(\mathcal{Z},\mathcal{Q},\mathcal{L})$ such that the action of $\mathbb{G}$ commutes with the action $\sigma_\eta$ generated by $\eta$ and the action of $\mathbb{G}$ on $(\mathcal{Z},\mathcal{Q},\mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \overset{\mathrm{i}_\eta}{\cong} (Z, Q, L) \times \mathbb{C}^*$ coincides with the action of $\mathbb{G}$ on (the first factor of) $(Z, Q, L) \times \mathbb{C}^*$.*

**Definition 3.2** ([39])**.** *For any $\xi \in N_{\mathbb{R}}$, the $\xi$-twist of $(\mathcal{Z},\mathcal{Q},\mathcal{L},\eta)$ is the data $(\mathcal{Z},\mathcal{Q},\mathcal{L},\eta+\xi)$, which, for simplicity, will also be denoted by $(\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi)$. If $\xi \in N_{\mathbb{Z}}$, then $(\mathcal{Z}_s,\mathcal{Q}_\xi,\mathcal{L}_\xi) = (\mathcal{Z},\mathcal{Q},\mathcal{L},\eta+\xi)$ is a test configuration. In general, we shall call $(\mathcal{Z},\mathcal{Q},\mathcal{L},\eta+\xi)$ to be an $\mathbb{R}$-test configuration.*

The twists of test configurations first appeared in the work of Hisamoto ([38, 39]). The following result begins to study the twists of test configurations from non-Archimedean point of view.

**Proposition 3.3.** *Let $(\mathcal{Z},\mathcal{Q},\mathcal{L})$ be a $\mathbb{G}$-equivariant dominating test configuration of $(Z, Q, L)$. For any $\xi \in N_{\mathbb{Z}}$, the non-Archimedean metric $\phi_{(\mathcal{Z}_\xi,\mathcal{L}_\xi)}$ defined by the twisted test configuration is related to $\phi_{(\mathcal{Z},\mathcal{L})}$ by the following identity: for any $w \in \mathring{\mathrm{Val}}(Z)$*

$$\phi_{(\mathcal{Z}_\xi,\mathcal{L}_\xi)}(w) = \phi_{(\mathcal{Z},\mathcal{L})}(w_\xi) + \theta_\xi^L(w), \tag{96}$$

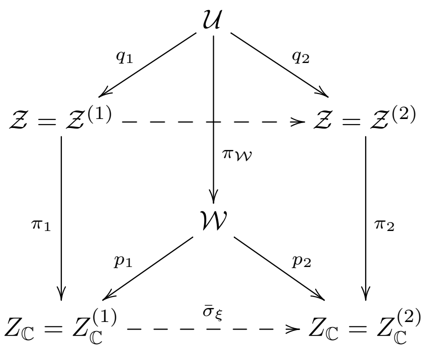
*where the function $\theta_\xi^L$, also denoted by $\theta_\xi$ if the $\mathbb{T}$-equivariant $\mathbb{Q}$-line bundle $L = -K_Z - Q$ is clear, is given by:*

$$\theta_\xi(w) = A_{(Z,Q)}(w_\xi) - A_{(Z,Q)}(w). \tag{97}$$

*Moreover, the following identities hold true:*

$$\begin{aligned}\mathbf{E}^{\mathrm{NA}}(\mathcal{Z}_\xi,\mathcal{L}_\xi) &= \mathbf{E}^{\mathrm{NA}}(\mathcal{Z},\mathcal{L}) - \mathrm{Fut}_{(Z,Q)}(\xi); &(98)\\ \mathbf{L}^{\mathrm{NA}}(\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi) &= \mathbf{L}^{\mathrm{NA}}(\mathcal{Z},\mathcal{Q},\mathcal{L}); &(99)\\ \mathbf{D}^{\mathrm{NA}}(\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi) &= \mathbf{D}^{\mathrm{NA}}(\mathcal{Z},\mathcal{Q},\mathcal{L}) - \mathrm{Fut}_{(Z,Q)}(\xi). &(100)\end{aligned}$$

*Proof.* Since $\hat{\sigma}_\xi(t)$ be the $\mathbb{C}^*$-action generated by $\xi$, we can let $\bar{\sigma}_\xi : Z_{\mathbb{C}} \dashrightarrow Z_{\mathbb{C}}$ be the birational map given by for any $(x,t) \in Z \times \mathbb{C}^*$: $(x,t) \mapsto (\hat{\sigma}_\xi(t) \circ x, t)$. Consider the commutative diagram:




(101)

The map $\pi_1 \circ q_1$ is $\eta$-equivariant. Moreover, the test configuration $(\mathcal{Z}_\xi, \mathcal{L}_\xi)$ is equivalent to the test configuration $(\mathcal{U}, q_2^*\mathcal{L}, \eta)$. We now decompose:

$$\begin{aligned} q_2^*\mathcal{L} - q_1^*\pi_1^* L_{\mathbb{C}} &= q_2^*\mathcal{L} - q_2^*\pi_2^* L_{\mathbb{C}} + q_2^*\pi_2^* L_{\mathbb{C}} - q_1^*\pi_1^* L_{\mathbb{C}} \\ &= q_2^*(\mathcal{L} - \pi_2^* L_{\mathbb{C}}) + \pi_{\mathcal{W}}^*(p_2^* L_{\mathbb{C}} - p_1^* L_{\mathbb{C}}). \end{aligned} \tag{102}$$

For any $w \in \mathring{\mathrm{Val}}(Z)$, for any $f \in \mathbb{C}(Z)_\alpha$, let $\bar{f} = p_1^* f$ denote the function on $Z \times \mathbb{C}^*$ via the projection $p_1$ to the first factor. Then $\bar{\sigma}_\xi^* \bar{f} = t^{\langle \alpha, \xi \rangle} \bar{f}$. By the definition of Gauss extension, we get:

$$\begin{aligned} (q_2)_* G(w)(\bar{f}) &= G(w)((q_2)^* \bar{f}) = G(w)(t^{\langle \alpha, \xi \rangle} \bar{f}) = \langle \alpha, \xi \rangle + G(w)(\bar{f}) \\ &= G(w_\xi)(\bar{f}). \end{aligned}$$

So $(q_2)_* G(w) = G(w_\xi)$. For any $w \in \mathring{\mathrm{Val}}(Z)$, by (102), we have:

$$\phi_\xi(w) = \phi(w_\xi) + \theta_\xi(w),$$

where $\theta_\xi(w) = G(w)(p_2^* L_{\mathbb{C}} - p_1^* L_{\mathbb{C}})$. To get identity (97), we calculate:

$$\begin{aligned} A_{(Z,Q)}(w) + \theta_\xi(w) &= A_{(Z_{\mathbb{C}}^{(1)}, Q_{\mathbb{C}}^{(1)})}(G(w)) - 1 - G(w)\left(p_2^*(K_{Z_{\mathbb{C}}^{(2)}} + Q_{\mathbb{C}}) - p_1^*(K_{Z_{\mathbb{C}}^{(1)}} + Q_{\mathbb{C}})\right) \\ &= G(w)\left(K_{\mathcal{W}/(Z_{\mathbb{C}}^{(2)}, Q_{\mathbb{C}}^{(2)})}\right) - 1 = (\bar{\sigma}_\xi)_* G(w)\left(K_{\mathcal{W}/(Z_{\mathbb{C}}^{(1)}, Q_{\mathbb{C}}^{(1)})}\right) - 1 \\ &= G(w_\xi)\left(K_{\mathcal{W}/(Z_{\mathbb{C}}, Q_{\mathbb{C}})}\right) - 1 = A_{(Z,Q)}(w_\xi). \end{aligned}$$

By (97) and (96), we have the identity:

$$A_{(Z,Q)}(w) + \phi_\xi(w) = A_{(Z,Q)}(w) + \phi(w_\xi) + \theta_\xi(w) = A_{(Z,Q)}(w_\xi) + \phi(w_\xi).$$

Taking the infimum over $w$ on both sides and by the change of variable, we get the identity (99).

Let us prove (98). Assume $\mathcal{L} = \pi^*(-K_Z - Q) + E$. Let $\mathcal{L}_b = \pi^*(-K_Z - Q) + bE$. Consider

$$h(b) := \frac{1}{n+1} \overline{q_2^* \mathcal{L}_b}^{\cdot n+1} - \frac{1}{n+1} \overline{q_1^* \mathcal{L}_b}^{\cdot n+1},$$

where the compactifications we use are using the isomorphism induced by $\eta$.

$$\frac{b}{db} h(b) = q_2^* \mathcal{L}_b^{\cdot n} \cdot q_2^* E - q_1^* \mathcal{L}_b^{\cdot n} \cdot q_1^* E = 0.$$

So we get:

$$\begin{aligned}\mathbf{E}^{\mathrm{NA}}(\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi)-\mathbf{E}^{\mathrm{NA}}(\mathcal{Z},\mathcal{Q},\mathcal{L}) &= \frac{1}{n+1}\overline{q_2^*\mathcal{L}}^{\cdot n+1}-\frac{1}{n+1}\overline{q_1^*\mathcal{L}}^{\cdot n+1}=h(1)=h(0)\\ &= \frac{1}{n+1}\overline{q_2^*L}^{\cdot n+1}-\frac{1}{n+1}\overline{q_1^*L}^{\cdot n+1}\\ &= \mathrm{CW}_L(\xi)=-\mathrm{Fut}_{(Z,Q)}(\xi). \quad \text{(see (73))}\end{aligned}$$

The identity (100) follows from (99) and (98).

□

**Remark 3.4.** *Note that the identities (98)-(100) by using Archimedean energy functionals. Let $\Phi=\{\varphi(t)\}$ be a smooth and positively curved Hermitian metric on $\mathcal{L}$. Then $\hat{\sigma}_\xi(t)^*\Phi:=\{\hat{\sigma}_\xi(t)^*\varphi(t)\}$ is a smooth and positively curved Hermitian metric on $(\mathcal{Z}_\xi,\mathcal{L}_\xi)$. On the other hand, because the action of $\mathbb{T}\cong(\mathbb{C}^*)^r$ on $-(K_Z+Q)$ is induced by the pull back of (logarithmic) $n$-forms, one can easily verify that:*

$$\mathbf{L}(\hat{\sigma}_\xi(t)^*\varphi(t))=\mathbf{L}(\varphi(t)),\quad \mathbf{E}(\hat{\sigma}_\xi(t)^*\varphi(t))=\mathbf{E}(\varphi(t))-\log|t|^2\cdot\mathrm{Fut}(\xi).$$

*The identities (98)-(99) follow by taking the slope at infinity and using (48).*

If $\xi\in N_{\mathbb{Q}}$ and $b\xi\in N_{\mathbb{Z}}$ for some $b\in\mathbb{N}$, then $(\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi)$ induces a test configuration by base change:

$$(\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi)^{(b)}:=\left(\text{ normalization of }(\mathcal{Z},\mathcal{Q},\mathcal{L})\times_{\mathbb{C},\mathrm{m}_b}\mathbb{C},b\eta+b\xi\right), \tag{103}$$

where $\mathrm{m}_b:t'\to t'^b=t$, $b\eta:=b\cdot\mathrm{m}_b^*\eta$ and $b\xi=b\cdot\mathrm{m}_b^*\xi$. Then with $\phi=\phi_{(\mathcal{Z},\mathcal{L})}$, we define the $\xi$-twist of $\phi$ to be the non-Archimedean metric represented by the following function on $\mathring{\mathrm{Val}}(Z)$:

$$\phi_\xi(v)=(b^{-1}\circ\phi_{(\mathcal{Z}_\xi,\mathcal{L}_\xi)^{(b)}})(v). \tag{104}$$

For the non-Archimedean energies appearing in (41)-(46), we also set:

$$\mathbf{F}^{\mathrm{NA}}(\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi)=b^{-1}\mathbf{F}^{\mathrm{NA}}((\mathcal{Z}_\xi,\mathcal{Q}_\xi,\mathcal{L}_\xi)^{(b)}). \tag{105}$$

**Lemma 3.5.** *For any $\xi\in N_{\mathbb{Q}}$, the same identity as in (96) holds true:*

$$\phi_\xi(v)=\phi(v_\xi)+\theta_\xi(v). \tag{106}$$

*Proof.* For simplicity, we write $\phi_{(\mathcal{Z},\mathcal{L})^{(b)}}=b\circ\phi$. From (104) and (96), we can calculate:

$$\begin{aligned}\phi_\xi(v) &= (b^{-1}\circ(b\circ\phi)_{b\xi})(v)=b^{-1}\cdot(b\circ\phi)_{b\xi}(bv)\\ &= b^{-1}\cdot((b\circ\phi)((bv)_{b\xi})+\theta_{b\xi}(bv))\\ &= b^{-1}\cdot\left(b\cdot\phi(b^{-1}(bv)_{b\xi})+\theta_{b\xi}(bv)\right)\\ &= \phi(v_\xi)+b^{-1}\theta_{b\xi}(bv).\end{aligned}$$

Now we can note that:

$$\begin{aligned}b^{-1}\theta_{b\xi}(bv) &= b^{-1}\left(A_{(Z,Q)}((bv)_{b\xi})-A_{(Z,Q)}(bv)\right)\\ &= A_{(Z,Q)}(v_\xi)-A_{(Z,Q)}(v)=\theta_\xi(v).\end{aligned}$$

□

For any $\xi\in N_{\mathbb{R}}$, we can define $\phi_\xi$ using the formula (106). We will see in the following subsection that the twist $\phi_\xi$ can be understood as non-Archimedean metrics from twisted filtrations. Indeed, the identity (106) is nothing but the non-Archimedean analogue of the well-known formula in the Archimedean case.

## 3.2 Twists of filtrations

Let $\mathcal{F} = \mathcal{F}R_\bullet$ be a filtration of $R = R^{(\ell_0)} = \bigoplus_{m=0}^{+\infty} H^0(Z, m\ell_0 L)$. Assume that $\mathcal{F}$ is $\mathbb{T}$-equivariant, which means that $\mathcal{F}^x R_m$ is a $\mathbb{T}$-invariant subspace of $R_m$ for any $x \in \mathbb{R}$. For $\alpha \in M_\mathbb{Z} = N_\mathbb{Z}^\vee$, denote the weight space

$$(R_m)_\alpha = \{s \in R_m; \tau \circ s = \tau^\alpha s \text{ for all } \tau \in (\mathbb{C}^*)^r\}. \tag{107}$$

Then we have:

$$(\mathcal{F}^x R_m)_\alpha := \{s \in \mathcal{F}^x R_m; \tau \circ s = \tau^\alpha s\} = \mathcal{F}^x R_m \cap (R_m)_\alpha, \tag{108}$$

and the decomposition:

$$\mathcal{F}^x R_m = \bigoplus_{\alpha \in M_\mathbb{Z}} (\mathcal{F}^x R_m)_\alpha. \tag{109}$$

**Definition 3.6.** *For any $\xi \in N_\mathbb{R}$, the $\xi$-twist of $\mathcal{F}$ is the filtration $\mathcal{F}_\xi R_\bullet$ defined by:*

$$\mathcal{F}_\xi^x R_m = \bigoplus_{\alpha \in M_\mathbb{Z}} (\mathcal{F}_\xi^x R_m)_\alpha, \quad \textit{where} \quad (\mathcal{F}_\xi^x R_m)_\alpha := (\mathcal{F}^{x - \langle \alpha, \xi \rangle} R_m)_\alpha. \tag{110}$$

**Example 3.7.** *Let $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ be a test configuration of $(Z, Q, L)$, which determines a filtration $\mathcal{F} := \mathcal{F}_{(\mathcal{Z}, \ell_0 \mathcal{L})}$ of $R^{(\ell_0)}$ (see Example 2.24). Recall that $s \in \mathcal{F}^x R_m$ if and only if $t^{-\lceil x \rceil} \bar{s}$ extends to a holomorphic section. Let $\xi \in N_\mathbb{Z}$. If $s \in (\mathcal{F}^x R_m)_\alpha$, then $\bar{\sigma}_\xi^* \bar{s} = t^{\langle \alpha, \xi \rangle} \bar{s}$ which implies $s \in \left(\mathcal{F}^{x - \langle \alpha, \xi \rangle}_{(\mathcal{Z}_\xi, \ell_0 \mathcal{L}_\xi)} R_m\right)_\alpha$. So we get the identification: $\mathcal{F}^x_{(\mathcal{Z}_\xi, \ell_0 \mathcal{L}_\xi)} R_m = \mathcal{F}^x_{(\mathcal{X}, \ell_0 \mathcal{L}), \xi} R_m$.*

The following proposition deals with twists of filtrations associated to valuations.

**Proposition 3.8.** *Let $v \in \mathring{\mathrm{Val}}(Z)^\mathbb{T}$ and $\mathcal{F} = \mathcal{F}_v$ be as defined in (67). We have the following identification of the filtration associated to the twisted valuation: for any $\xi \in N_\mathbb{R}$*

$$(\mathcal{F}^x_{v_\xi} R_m)_\alpha = \left(\mathcal{F}_v^{x - \langle \alpha, \xi \rangle - m \ell_0 \theta_\xi(v)} R_m\right)_\alpha, \tag{111}$$

*where $\theta_\xi(v) = \theta_\xi^L(v)$ is given by (97):*

$$\theta_\xi(v) = A_{(Z,Q)}(v_\xi) - A_{(Z,Q)}(v). \tag{112}$$

*Proof.* Let $W = \mathrm{center}(v)$ (resp. $W'$) be the center of $v$ (resp. $v_\xi$) on $Z$. Let $U$ (resp. $U'$) be a $\mathbb{T}$-invariant Zariski open set such that $U \cap W \neq \emptyset$ (resp. $U' \cap W' \neq \emptyset$), and let $\mathfrak{e}$ (resp. $\mathfrak{e}'$) be an equivariant nonvanishing section of $-\ell_0(K_Z + Q)|_U$ (resp. $-\ell_0(K_Z + Q)|_{U'}$).

Assume $s \in (\mathcal{F}^x_{v_\xi} R_m)_\alpha$. Write $s = f \mathfrak{e}^m$ on $U$ and $s = f' \mathfrak{e}'^m$ on $U'$ for $f \in \mathcal{O}_Z(U)$ and $f' \in \mathcal{O}_Z(U')$. We have the identity:

$$\langle \alpha, \xi \rangle = \frac{\mathscr{L}_\xi s}{s} = \frac{\mathscr{L}_\xi(f)}{f} + m \frac{\mathscr{L}_\xi \mathfrak{e}}{\mathfrak{e}}.$$

Then we have the following identities:

$$\begin{aligned} v_\xi(s) &= v_\xi(f') = v(f') + \frac{\mathscr{L}_\xi f'}{f'} \\ &= v(f) + v\left(\frac{\mathfrak{e}^m}{\mathfrak{e}'^m}\right) + \langle \alpha, \xi \rangle - m \frac{\mathscr{L}_\xi \mathfrak{e}'}{\mathfrak{e}'} \\ &= v(s) + \langle \alpha, \xi \rangle + m \left( v\left(\frac{\mathfrak{e}}{\mathfrak{e}'}\right) - \frac{\mathscr{L}_\xi \mathfrak{e}'}{\mathfrak{e}'} \right) \\ &= v(s) + \langle \alpha, \xi \rangle + \ell_0 m \cdot \tilde{\theta}_\xi(v), \end{aligned} \tag{113}$$

where
$$\tilde{\theta}_\xi(v) = \frac{1}{\ell_0}\left(v\left(\frac{\mathfrak{e}}{\mathfrak{e}'}\right) - \frac{\mathscr{L}_\xi \mathfrak{e}'}{\mathfrak{e}'}\right) =: \frac{1}{\ell_0}\left(v\left(\frac{\mathfrak{e}}{\mathfrak{e}'}\right) - \mathbf{c}\right). \tag{114}$$

So $v_\xi(s) \geq x$ if and only if $v(s) \geq x - \langle \alpha, \xi\rangle - \tilde{\theta}_\xi(v)$. We need to verify $\tilde{\theta}_\xi = \theta_\xi$. To see this, we use the commutative diagram and calculate.

$$\begin{aligned}
\theta_\xi(v) &= -G(v)(p_2^* L_{\mathbb{C}} - p_1^* L_{\mathbb{C}}) = G(v)(p_2^*((K_Z + Q) \times \mathbb{C}) - p_1^*((K_Z + Q) \times \mathbb{C}) \\
&= -\frac{1}{\ell_0} G(v)\left(\frac{p_2^* \bar{\mathfrak{e}}'}{p_1^* \bar{\mathfrak{e}}}\right) = -\frac{1}{\ell_0} G(v)\left(\frac{p_1^* \bar{\sigma}_\xi^* \bar{\mathfrak{e}}'}{p_1^* \bar{\mathfrak{e}}}\right) = \frac{1}{\ell_0}\left(-G(v)\left(\frac{p_1^* \bar{\sigma}_\xi^* \bar{\mathfrak{e}}'}{p_1^* \bar{\mathfrak{e}}'}\right) - G(v)\left(\frac{p_1^* \bar{\mathfrak{e}}'}{p_1'^* \bar{\mathfrak{e}}}\right)\right) \\
&= -\frac{1}{\ell_0}\left(G(v)\left(t^{\mathbf{c}}\right) - v\left(\frac{\mathfrak{e}'}{e}\right)\right) = \frac{1}{\ell_0}\left(v\left(\frac{\mathfrak{e}}{\mathfrak{e}'}\right) - \mathbf{c}\right) = \tilde{\theta}_\xi(v).
\end{aligned}$$

□

**Proposition 3.9.** *Let $\mathcal{F}$ be a $\mathbb{T}$-equivariant filtration and $\xi \in N_{\mathbb{R}}$. For any $w \in \mathring{\mathrm{Val}}(Z)^{\mathbb{T}}$, we have the following identities:*

$$\phi_m^{\mathcal{F}_\xi}(w) = \phi_m^{\mathcal{F}}(w_\xi) + \theta_\xi(w) \tag{115}$$
$$\phi^{\mathcal{F}_\xi}(w) = \phi^{\mathcal{F}}(w_\xi) + \theta_\xi(w). \tag{116}$$

*Proof.* Note that the second identity is obtained from the first one by letting $m \to +\infty$. So we just need to prove the first identity. Set

$$(I_{m,x}^{\mathcal{F}_\xi})_\alpha = \mathrm{Im}\left((\mathcal{F}^x R_m)_\alpha \otimes \mathcal{O}_Z(m\ell_0 L) \to \mathcal{O}_Z\right). \tag{117}$$

By definitions in (56) and (110), we have an identity of ideals:

$$(I_{m,x}^{\mathcal{F}_\xi})_\alpha = (I_{m,x-\langle\alpha,\xi\rangle}^{\mathcal{F}})_\alpha \tag{118}$$

So by (57) we have identities of fractional ideals:

$$\tilde{\mathcal{I}}_m^{\mathcal{F}} = \sum_x \sum_\alpha (I_{m,x}^{\mathcal{F}})_\alpha t^{-x}, \quad \tilde{\mathcal{I}}_m^{\mathcal{F}_\xi} = \sum_x \sum_\alpha (I_{m,x-\langle\alpha,\xi\rangle}^{\mathcal{F}})_\alpha t^{-x} \tag{119}$$

Applying the definition non-Archimedean metric associated to filtrations in (87) to $\phi^{\mathcal{F}_\xi}$ and using the $\mathbb{C}^* \times \mathbb{T}$-invariance of the valuation of any $G(w)$, we indeed get (115):

$$\begin{aligned}
-\phi_m^{\mathcal{F}_\xi}(w) &= \frac{1}{m\ell_0} \min_\alpha \min_x \left(w((I_{m,x}^{\mathcal{F}_\xi})_\alpha) - x\right) \\
&= \frac{1}{m\ell_0} \min_\alpha \min_x \left(w((I_{m,x-\langle\alpha,\xi\rangle}^{\mathcal{F}})_\alpha) - x\right) \\
&= \frac{1}{m\ell_0} \min_\alpha \min_x \left(w((I_{m,x}^{\mathcal{F}})_\alpha) - x - \langle\alpha,\xi\rangle\right) \\
&= -\theta_\xi(w) - \frac{1}{m\ell_0} \min_\alpha \min_x \left(w_\xi((I_{m,x}^{\mathcal{F}})_\alpha) - x\right) \quad \text{(by (113))} \\
&= -\theta_\xi(w) - \phi_m^{\mathcal{F}}(w_\xi).
\end{aligned}$$

□

**Lemma 3.10.** *For any $\xi \in N_{\mathbb{R}}$, the following identities hold true:*

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}); \tag{120}$$
$$\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{E}^{\mathrm{NA}}(\mathcal{F}) - \mathrm{Fut}_{(Z,Q)}(\xi); \tag{121}$$
$$\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{D}^{\mathrm{NA}}(\mathcal{F}) - \mathrm{Fut}_{(Z,Q)}(\xi). \tag{122}$$

*In particular, if $\mathrm{Fut}_{(Z,Q)} \equiv 0$, then $\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{E}^{\mathrm{NA}}(\mathcal{F})$ and $\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{D}^{\mathrm{NA}}(\mathcal{F})$.*

*Proof.* By (116) and (112), we get

$$A_{(Z,Q)}(v) + \phi_\xi(v) = A_{(Z,Q)}(v) + \phi(v_\xi) + \theta_\xi(v) = A_{(Z,Q)}(v_\xi) + \phi(v_\xi). \tag{123}$$

Taking infimum for $v$ ranging in $\mathring{\mathrm{Val}}$ we get the identity (120).

Next choose a basis $\{s_1^{(m)}, \dots, s_{N_m}^{(m)}\}$ adapted to the filtration $\{\mathcal{F}^x R_m\}$, which means that

$$\mathcal{F}^x R_m = \mathrm{span}\{s_1^{(m)}, \dots, s_{k_x}^{(m)}\} \tag{124}$$

for some $k_x \in \{1, \dots, N_m\}$. Because $\mathcal{F}^x R_m$ is $(\mathbb{C}^*)^r$-invariant, we can assume that $s_j^{(m)}$ are equivariant in the sense that:

$$\tau \circ s_j^{(m)} = \tau^{\alpha_j^{(m)}} \cdot s_j^{(m)}. \tag{125}$$

Let $\lambda_1^{(m)} \geq \lambda_2^{(m)} \cdots \geq \lambda_{N_m}^{(m)}$ be the succesive minima. Because of the $\mathbb{T}$-equivariance,

$$\lambda_j^{(m)} + \langle \alpha_j^{(m)}, \xi \rangle =: \lambda_j^{(m)} + \kappa_j^{(m)}, \quad j = 1, \dots, N_m, \tag{126}$$

are the set of successive minima for the twisted filtration. So we get:

$$\begin{aligned} \mathbf{E}^{\mathrm{NA}}(\mathcal{F}_\xi) &= \frac{1}{N_m} \lim_{m \to +\infty} \sum_{j=1}^{N_m} \frac{\lambda_j^{(m)} + \kappa_j^{(m)}}{m\ell_0} \\ &= \mathbf{E}^{\mathrm{NA}}(\mathcal{F}) + \mathrm{CW}_L(\xi). \end{aligned} \tag{127}$$

Finally recall that In our set-up, $\mathrm{CW}_L(\xi) = -\mathrm{Fut}_{(Z,Q)}(\xi)$ (see (73)).

□

**Definition 3.11.** *For any $v \in \mathring{\mathrm{Val}}(Z)$, define the invariant:*

$$\beta(v) := \beta_{(Z,Q)}(v) = A_{(Z,Q)}(v) - S_L(v). \tag{128}$$

**Proposition 3.12.** *For any $v \in \mathring{\mathrm{Val}}(Z)$ we have the inequality:*

$$\beta(v) \geq \mathbf{D}^{\mathrm{NA}}(\mathcal{F}_v). \tag{129}$$

*Moreover for any $\xi \in N_{\mathbb{R}}$, we have the identity:*

$$\beta(v_\xi) = \beta(v) - \mathrm{Fut}_{(Z,Q)}(\xi). \tag{130}$$

*Proof.* Recall that

$$\mathbf{D}(\mathcal{F}_v) = \mathbf{D}(\phi_v) = -\mathbf{E}^{\mathrm{NA}}(\phi_v) + \mathbf{L}^{\mathrm{NA}}(\phi_v). \tag{131}$$

By (69), we have

$$S_L(v) \quad = \quad \mathbf{E}^{\mathrm{NA}}(\mathcal{F}_v) = \frac{1}{\ell_0^n L^{\cdot n}} \int_0^{+\infty} -\frac{x}{\ell_0} \cdot d\,\mathrm{vol}(\mathcal{F}^{(x)} R^{(\ell_0)})$$

Moreover, since $\phi_v(v) = 0$ (by Lemma 2.32),

$$\mathbf{L}^{\mathrm{NA}}(\phi_v) = \inf_w (A(w) + \phi_v(w)) \leq A(v). \tag{132}$$

So we get (129). Because by (111) $\mathcal{F}_{v_\xi} = \mathcal{F}_\xi(\theta_\xi(v))$ (see (51)), we use (121) and (112) to get the identity (130):

$$\begin{aligned} S_L(v_\xi) &= \mathbf{E}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) = \mathbf{E}^{\mathrm{NA}}(\mathcal{F}_\xi(\theta_\xi(v)) \\ &= \mathbf{E}^{\mathrm{NA}}(\mathcal{F}_v) + \mathrm{Fut}_{(Z,Q)}(\xi) + \theta_\xi(v) \\ &= S_L(v) + \mathrm{Fut}_{(Z,Q)}(\xi) + A(v_\xi) - A(v). \end{aligned}$$

□

### 3.3 $\mathbb{G}$-Uniform Ding stability

Let $(Z, Q)$, $L = -K_Z - Q$, $\mathbb{G}$ and $\mathbb{T}$ be as before.

**Definition 3.13.** *For any $\mathbb{T}$-equivariant test configuration $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ of $(Z, Q, L)$, the reduced $\mathbf{J}$-norm of $(\mathcal{Z}, \mathcal{L})$ is defined as:*

$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{Z}, \mathcal{L}) = \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{Z}_\xi, \mathcal{L}_\xi). \tag{133}$$

*For any graded filtration $\mathcal{F}$, its reduced $J$-norm is defined as:*

$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{F}) = \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi). \tag{134}$$

The reason for defining $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}$ comes from Hisamoto's slope formula:

**Theorem 3.14** ([39, Theorem B]). *Let $(\mathcal{Z}, \mathcal{L})$ be a $\mathbb{T}$-equivariant ample normal test configuration of $(Z, L)$. Let $\Phi = \{\varphi(s); s = -\log|t| \in [0, +\infty)\}$ be a bounded positively curved Hermitian metric on $\mathcal{L}$. Then we have the following limit formula:*

$$\lim_{s \to +\infty} \frac{\mathbf{J}_{\mathbb{T}}(\varphi(s))}{2s} = \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{Z}, \mathcal{L}). \tag{135}$$

Since Hisamoto works on smooth manifold, for the convenience of the reader, we provide a proof of this result essentially following the argument in [39] (which builds on some ideas of Berman). This will show that the arguments indeed work for any normal projective varieties.

*Proof of 3.14.* We can assume that $\Phi$ is a locally bounded weak geodesic ray. Then one can easily check that $(s, \xi) \mapsto \mathbf{J}(\sigma_\xi(s)^*\varphi(s))$ is a convex function on $\mathbb{R} \times N_{\mathbb{R}}$ (see Proposition 5.1). So it is easy to see that $f(s) := \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}(\sigma_\xi(s)^*\varphi(s))$ is a convex function on $s$. So the limit in (135) exists. Denote left and right sides of (135) by $a$ and $b$ respectively. Using the slope formula for $\mathbf{J}$, it is easy to verify the inequality "$a \le b$" in (135). For the other direction, by Lemma 2.12, there exists $\xi_s \in N_{\mathbb{R}}$ such that $\mathbf{J}_{\mathbb{T}}(\varphi(s)) = \mathbf{J}(\sigma_{\xi_s}(s)^*\varphi(s))$. By the quasi-triangle inequality for $\mathbf{I}$ ([6, Theorem 1.8]) and hence for $\mathbf{J}$, we have (for any fixed reference metric $\psi$):

$$\begin{aligned} \mathbf{J}_\psi(\sigma_{\xi_s}(s)^*\psi) &\le c_n \left(\mathbf{J}_\psi(\sigma_{\xi_s}(s)^*\varphi(s)) + \mathbf{J}_{\sigma_{\xi_s}(s)^*\varphi(s)}(\sigma_{\xi_s}(s)^*\psi\right) \\ &\le C\mathbf{J}_\psi(\varphi(s)) = C(\mathbf{J}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L})s + o(s)) \le C's. \end{aligned}$$

By the properness of $\xi \mapsto \mathbf{J}(\sigma_\xi(1)^*\psi)$ (Lemma 2.12) and the identity $\sigma_{\xi_s}(s) = \sigma_{s\xi_s}(1)$, this means that $\xi_s$ is uniformly bounded in $N_{\mathbb{R}}$. Hence there exists $s_\infty \in N_{\mathbb{R}}$ and a sequence $s_j \to +\infty$ such that $\xi_{s_j} \to \xi_\infty$. We just need to show that

$$\lim_{j \to +\infty} s_j^{-1} \left| \mathbf{J}_\psi(\sigma_{\xi_{s_j}}(s_j)^*\varphi(s_j)) - \mathbf{J}_\psi(\sigma_{\xi_\infty}(s_j)^*\varphi(s_j)) \right| = 0, \tag{136}$$

since it would imply:

$$\begin{aligned} a = \lim_{j \to +\infty} \frac{\mathbf{J}(\sigma_{\xi_{s_j}}(s_j)^*\varphi(s_j))}{2s_j} &= \lim_{s \to +\infty} \frac{\mathbf{J}(\sigma_{\xi_\infty}(s_j)^*\varphi(s_j))}{2s_j} \\ &= \mathbf{J}^{\mathrm{NA}}(\mathcal{Z}_{\xi_\infty}, \mathcal{L}_{\xi_\infty}) \ge \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{Z}, \mathcal{L}) = b. \end{aligned}$$

To verify (136), we use the easy fact $|\mathbf{J}(\varphi_1) - \mathbf{J}(\varphi_2)| \le 2\sup_X |\varphi_1 - \varphi_2|$ to reduce to showing:

$$\lim_{j \to +\infty} s_j^{-1} \sup_X |\sigma_{\xi_{s_j}}(s_j)^*\varphi(s_j) - \sigma_{\xi_\infty}(s_j)^*\varphi(s_j)| = 0. \tag{137}$$

Now we fix a $\mathbb{C}^* \times \mathbb{T}$-equivariant embedding $\iota : \mathcal{X} \to \mathbb{P}^{N_k-1} \times \mathbb{C}$ with that $\iota^*\mathcal{O}_{\mathbb{P}^{N_k-1}}(1) = \mathcal{L}^k$. The weight decomposition of $H^0(X, kL)$ allows us to choose homogeneous coordinates $\{Z_1, \dots, Z_{N_k}\}$ on $\mathbb{P}^{N_k-1}$ such that the $\mathbb{C}^* \times \mathbb{T}$-action is given by:

$$(\tau_0, \tau_1, \dots, \tau_r) \cdot Z_i = \tau_0^{\lambda_i} \prod_{p=1}^{r} \tau_k^{\alpha_i^p} \cdot Z_i. \tag{138}$$

Identify $X$ with the fibre at $t = 1$: $X \cong p_2^{-1}(\{1\}) \cap \mathcal{X}$, and set $e^{-\psi_{\mathrm{FS}}} = \iota^* h_{\mathrm{FS}}^{1/k}\Big|_X$ where $h_{\mathrm{FS}}$ is the standard Fubini-Study metric on $\mathbb{P}^{N_k-1}$. Then to verify (136), we can replace the weak geodesic ray $\{\varphi(s)\}$ by the $L^\infty$-comparable $\{\tilde{\varphi}(s) = \sigma_\eta(s)^*\psi_{\mathrm{FS}}\}$, which is given by the well-known explicit formula (recall that $s = -\log|t|$):

$$\tilde{\varphi}(s) - \psi_{\mathrm{FS}} = \frac{1}{k} \log \frac{\sum_{i=1}^{N_k} |t|^{-2\lambda_i} |Z_i|^2}{\sum_{i=1}^{N_k} |Z_i|^2}.$$

More generally, for any $\xi \in N_{\mathbb{R}}$, $\sigma_\xi(s)^*\tilde{\varphi}(s)$ is given by:

$$\sigma_\xi(s)^*\tilde{\varphi}(s) - \psi_{\mathrm{FS}} = \frac{1}{k} \log \frac{\sum_i |t|^{-2(\lambda_i + \langle \alpha_i, \xi \rangle)} |Z_i|^2}{\sum_i |Z_i|^2}.$$

So we easily get for any $\xi, \xi' \in N_{\mathbb{R}}$ (again with $s = -\log|t|$),

$$|\sigma_\xi(s)^*\tilde{\varphi}(s) - \sigma_{\xi'}(s)^*\tilde{\varphi}(s)| = \frac{1}{k} \left| \log \frac{\sum_i |t|^{-2(\lambda_i + \langle \alpha_i, \xi \rangle)} |Z_i|^2}{\sum_i |t|^{-2(\lambda_i + \langle \alpha_i, \xi' \rangle)} |Z_i|^2} \right| \le C(\log|t|^2)|\xi - \xi'|, \tag{139}$$

where $C = C(k, \{\alpha_i\})$ does not depend on $\xi, \xi', s$. Substituting the variables $\xi$, $\xi', s$ by $\xi_{s_j}, \xi_\infty, s_j$ respectively into the above estimate, we easily get the limit (137) by using the fact that $\xi_{s_j} \to \xi_\infty$. □

The next lemma generalizes [40, Lemma 3.18]:

**Lemma 3.15.** *Assume* $\mathrm{CW}_L \equiv 0$ *on* $\mathfrak{t}$. *Then for any* $\mathbb{T}$*-equivariant filtration* $\mathcal{F}$ *(satisfying the properties in Definition 2.21),* $\xi \mapsto \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi)$ *is a convex and proper function. More precisely, there exists* $C_1 > 0$ *depending only on the* $\mathbb{T}$*-action on* $Z$*, such that*

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi) \ge C_1|\xi| - (e_- + \mathbf{E}^{\mathrm{NA}}(\mathcal{F})), \tag{140}$$

*where* $e_-$ *is any number satisfying* $\mathcal{F}^{me_-} = 0$ *for* $m \in \mathbb{N}$ *(see Definition 2.21). As a consequence, it has a unique minimizer on* $N_{\mathbb{R}}$*. Moreover if* $\mathcal{F} = \mathcal{F}_{(\mathcal{Z}, \ell_0\mathcal{L})}$ *for some test configuration* $(\mathcal{Z}, \mathcal{L})$ *of* $(Z, L)$*, then the minimizer is contained in* $N_{\mathbb{Q}}$*.*

*Proof.* Assume that $m$ is sufficiently divisible such that $m\ell_0 L$ is globally generated. Let

$$\lambda_1^{(m)} \ge \lambda_2^{(m)} \ge \cdots \ge \lambda_{N_m}^{(m)} \tag{141}$$

be the successive minima of $\mathcal{F}R_m$. Then we have

$$\begin{aligned} \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi) &= \mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{F}_\xi) - \mathbf{E}^{\mathrm{NA}}(\mathcal{F}_\xi) \qquad \text{(see (61) – (62))} \\ &= \sup_m \max_j \frac{\lambda_j^{(m)} + \langle \alpha_j^{(m)}, \xi \rangle}{m\ell_0} - \mathbf{E}^{\mathrm{NA}}(\mathcal{F}) \quad \text{(by (121))} \qquad (142) \\ &\ge \max_j \frac{\langle \alpha_j^{(m)}, \xi \rangle}{m\ell_0} - (e_- + \mathbf{E}^{\mathrm{NA}}(\mathcal{F})). \qquad (143) \end{aligned}$$

The second identity used (127) and Proposition 2.22. The last inequality is because by definition 2.21 $\mathcal{F}$ is linearly bounded from below: $\lambda_j^{(m)} \geq m\ell_0 e_-$. From the expression (142) it is clear that $\xi \mapsto \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi) =: \mathbf{j}(\xi)$ is a convex function in $\xi \in N_{\mathbb{R}}$. We will show it is a proper function. Let $\mathbf{P} \subset M_{\mathbb{R}}$ be closed convex hull of the set:

$$\left\{ \frac{\alpha_j^{(m)}}{m\ell_0}; \quad j = 1, \ldots, N_m, m \in \mathbb{Z}_{\geq 0} \right\}. \tag{144}$$

The following measure is supported on $\mathbf{P}$.

$$\mathrm{DH}_{\mathbb{T}} = \lim_{m \to +\infty} \frac{1}{N_m} \sum_m \delta_{\frac{\alpha_j^{(m)}}{m\ell_0}}. \tag{145}$$

By [14, Proposition 6.4] (see also [18, Proposition 2.1] and [49]), $P$ is a rational polytope and $\mathrm{DH}_{\mathbb{T}}$ is absolutely continuous with respect to the Lebesgue measure. The Chow weight of $\xi$ is then given by:

$$\mathrm{CW}_L(\xi) = \lim_{m \to +\infty} \frac{1}{N_m} \sum_m \frac{\langle \alpha_j^{(m)}, \xi \rangle}{m\ell_0} = \int_{\mathbf{P}} \langle y, \xi \rangle \mathrm{DH}_{\mathbb{T}} = \mathrm{vol}(\mathbf{P}) \cdot \langle \mathrm{bc}_{\mathbb{T}}, \xi \rangle, \tag{146}$$

where $\mathrm{bc}_{\mathbb{T}}$ is the barycenter of $\mathrm{DH}_{\mathbb{T}}$.

If $\mathrm{CW} \equiv 0$ on $\mathfrak{t}$, then $\mathrm{bc}_{\mathbb{T}} = 0$. This implies that 0 is in the interior of $\mathbf{P}$. If $\Delta$ denotes the standard simplex, then there exists $\theta > 0$ such that $\theta\Delta \subset \mathbf{P}$. So for any $\epsilon > 0$ and $k = 1, \ldots, n$, there exist $m = m(\epsilon) \gg 1$ and $\alpha_{j_k^\pm}^{(m)}$, such that

$$\left| \frac{\alpha_{j_k^+}^{(m)}}{m\ell_0} - \theta \mathbf{e}_k \right| \leq \epsilon, \quad \left| \frac{\alpha_{j_k^-}^{(m)}}{m\ell_0} + \theta \mathbf{e}_k \right| \leq \epsilon. \tag{147}$$

So we get the inequality:

$$\left\langle \frac{\alpha_{j_k^\pm}^{(m)}}{m\ell_0}, \xi \right\rangle \geq \theta |\xi_k| - \epsilon |\xi|, \text{ for all } k. \tag{148}$$

Combining this with (143), we indeed get the properness of $\mathbf{j}(\xi)$:

$$\mathbf{j}(\xi) \geq \left( \frac{\theta}{\sqrt{n}} - \epsilon \right) |\xi| \tag{149}$$

Now assume $\mathcal{F} = \mathcal{F}_{(\mathcal{Z}, \ell_0 \mathcal{L})}$. When $m$ is sufficiently divisible such that $m\ell_0\mathcal{L}$ is globally generated, we have the identity:

$$\begin{aligned} \mathbf{J}^{\mathrm{NA}}(\mathcal{Z}_\xi, \mathcal{L}_\xi) &= \mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{Z}_\xi, \mathcal{L}_\xi) - \mathbf{E}^{\mathrm{NA}}(\mathcal{Z}_\xi, \mathcal{L}_\xi) \\ &= \max_j \frac{\lambda_j^{(m)} + \langle \alpha_j^{(m)}, \xi \rangle}{m\ell_0} - \mathbf{E}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}). \end{aligned} \tag{150}$$

We see that in this case $\mathbf{j}$ is a rationally piecewisely linear, convex and proper function on $N_{\mathbb{R}}$. So it obtains a minimum at some $\xi \in N_{\mathbb{Q}}$.

□

**Proposition 3.16.** *Assume* $\mathrm{CW}_L(\xi) \equiv 0$ *on* $N_{\mathbb{R}}$. *Let* $v \in \mathring{\mathrm{Val}}(X)^{\mathbb{T}}$, $\mathcal{F} = \mathcal{F}_v$ *and* $(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m)$ *be the* $m$*-th approximating test configurations of* $\mathcal{F}$ *as in Definition 2.23. Then we have:*

$$\limsup_{m \to +\infty} \mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m) = \mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\mathcal{F}). \tag{151}$$

*Proof.* By definition, we need to prove that:

$$\mathbf{I} := \limsup_{m\to+\infty} \inf_{\xi\in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) = \inf_{\xi\in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi) =: \mathbf{II}. \tag{152}$$

We first claim that for any $\xi \in N_{\mathbb{R}}$:

$$\lim_{m\to+\infty} \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) = \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi). \tag{153}$$

Indeed, by (115) we know $\phi_m^{\mathcal{F}_\xi} = \phi_{m,\xi}^{\mathcal{F}}$. On the other hand, by definition (see (88)) $\phi_{m,\xi}^{\mathcal{F}} = \phi_{(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m),\xi}$. So we get:

$$\mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) = \mathbf{J}^{\mathrm{NA}}\left(\phi_{(\check{\mathcal{Z}}_m,\check{\mathcal{L}}_m),\xi}\right) = \mathbf{J}^{\mathrm{NA}}(\phi_{m,\xi}^{\mathcal{F}}) = \mathbf{J}^{\mathrm{NA}}(\phi_m^{\mathcal{F}_\xi}). \tag{154}$$

So (153) follows from (78). (153) easily implies that $\mathbf{I} \le \mathbf{II}$, since for any $\xi \in N_{\mathbb{R}}$, we then have:

$$\limsup_{m\to+\infty} \inf_{\xi'\in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi'}, \check{\mathcal{L}}_{m,\xi'}) \le \lim_{m\to+\infty} \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) = \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi). \tag{155}$$

We only need to prove $\mathbf{II} \le \mathbf{I}$.

For simplicity of notations, set:

$$\begin{aligned} \mathbf{j}_m(\xi) &:= \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) = \mathbf{\Lambda}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) - \mathbf{E}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) \\ &=: \mathbf{\Lambda}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}) - \mathbf{E}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{L}}_m) =: \mathfrak{f}_m(\xi) + \mathfrak{g}_m(\xi). \\ \mathbf{j}(\xi) &:= \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{F}_\xi) - \mathbf{E}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{F}_\xi) - \mathbf{E}^{\mathrm{NA}}(\mathcal{F}) =: \mathfrak{f}(\xi) + \mathfrak{g}(\xi). \end{aligned}$$

Here we used (98), (121) and the assumption that $\mathrm{Fut}(\xi) = -\mathrm{CW}_L(\xi) = 0$.

By (77), we know that $\lim_{m\to+\infty} \mathfrak{g}_m = \mathfrak{g}$. By (140) from Lemma 3.15, we know that $\mathbf{j}_m(\xi)$ and $\mathbf{j}(\xi)$ satisfies the uniform properness estimates: there exist $C_1, C_2 > 0$ such that for any $\xi \in N_{\mathbb{R}}$, we have

$$\mathbf{j}_m(\xi) \ge C_1|\xi| - C_2, \quad \mathbf{j}(\xi) \ge C_1|\xi| - C_2. \tag{156}$$

So the infimum $\inf_{\xi\in N_{\mathbb{R}}} \mathbf{j}_m(\xi)$ and $\inf_{\xi\in N_{\mathbb{R}}} \mathbf{j}(\xi)$ are obtained on a uniformly bounded set of $\xi$, which we denote by $\Xi_{C_3} = \{\xi \in N_{\mathbb{R}}; |\xi| \le C_3\}$.

Moreover, by the proof of Lemma 3.15, $\mathfrak{f}_m$ and $\mathfrak{f}$ are all convex functions on $\mathbb{R}^r$. So $\mathfrak{f}_m$ are $\mathfrak{f}$ are continuous on $\mathbb{R}^r$. Choose $m_p := k^p, p \in \mathbb{N}$ for some $k \in \mathbb{N}$ sufficiently divisible. Then $\{\mathbf{j}_{m_p}\}_{p\in\mathbb{N}}$ is a monotone sequence of continuous functions converging pointwisely to $\mathbf{j}$ as $p \to +\infty$. By Dini's theorem, $\mathfrak{f}_{m_p}$ converges to $\mathfrak{f}$ uniformly on the compact set $\Xi_{C_3}$. By the above discussion, we know that as $p \to +\infty$, $\mathfrak{j}_{m_p}$ converges to $\mathfrak{j}$ uniformly over $\Xi_{C_3}$. So the convergence of infimum (over $\Xi_{C_3}$) also follows. □

**Remark 3.17.** *One can also use the uniform estimates from [11, section 5] to get uniform convergence over $\Xi_{C_3}$ in the above proof.*

**Definition 3.18** (see [39, 40])**.** *$(Z, Q)$ is $\mathbb{G}$-uniformly Ding-stable if there exists $\gamma > 0$ such that for any $\mathbb{G}$-equivariant test configuration $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ of $(Z, Q, L)$:*

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) \ge \gamma \cdot \mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{L}). \tag{157}$$

*If one replaces $\mathbf{D}^{\mathrm{NA}}$ by* CM*, then one gets the definition of $\mathbb{G}$-uniform K-stability.*

We should compare this notion with the following well-known definition:

**Definition 3.19.** *1. $(Z, Q)$ is $\mathbb{G}$-equivariantly uniformly Ding-stable if there exists $\gamma > 0$ such that for any $\mathbb{G}$-equivariant test configuration $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ of $(Z, Q, L)$:*

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) \ge \gamma \cdot \mathbf{J}(\mathcal{Z}, \mathcal{L}). \tag{158}$$

2. $(Z, Q)$ *is* $\mathbb{G}$*-equivariantly Ding-semistable if for any* $\mathbb{G}$*-equivariant test configuration* $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ *of* $(Z, Q, L)$*:*

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) \geq 0. \tag{159}$$

$(Z, Q)$ *is* $\mathbb{G}$*-equivariantly Ding-polystable if* $(Z, Q)$ *is* $\mathbb{G}$*-equivariantly Ding-semistable, and the identity in* (159) *holds only when* $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ *is a product test configuration.*

*If one replaces* $\mathbf{D}^{\mathrm{NA}}$ *by* CM *in the above definition, one gets the definition of* $\mathbb{G}$*-equivariantly uniform K-stability and so on.*

**Remark 3.20.** *By running* $\mathbb{C}^* \times \mathbb{G}$*-equivariant MMP, it is clear from the proof of [7, 33] (based on MMP process in [44]) that* $\mathbb{G}$*-equivariantly uniform Ding-stability is equivalent to* $\mathbb{G}$*-equivariantly uniform K-stability. The same remark applies to* $\mathbb{G}$*-equivariant semistability or polystability.*

Because $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}} \geq 0$, we see that $\mathbb{G}$-uniform Ding-stability implies that $\mathbb{G}$-equivariant Ding-semistability, which in particular implies $\mathrm{Fut}_{(Z,Q)} \equiv 0$ on $\mathfrak{t}$. In fact, $(Z, Q)$ is $\mathbb{G}$-uniformly Ding-stability implies that $(Z, Q)$ is $\mathbb{G}$-equivariant Ding-polystability:

**Lemma 3.21** ([38, 39])**.** *Assume* $\mathrm{CW}_L \equiv 0$ *on* $\mathfrak{t}$*. For any* $\mathbb{T}$*-equivariant test configuration* $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ *of* $(Z, Q, L)$*,* $\mathbf{J}_{\mathbb{T}}(\mathcal{Z}, \mathcal{L}) = 0$ *if and only if* $(\mathcal{Z}, \mathcal{L})$ *is a product test configuration generated by some* $\eta \in N_{\mathbb{Z}}$*. As a consequence, if* $(Z, Q)$ *is* $\mathbb{G}$*-uniformly Ding-stable, then for any* $\mathbb{G}$*-equivariant test configuration* $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ *of* $(Z, Q)$*,* $\mathbf{D}^{\mathrm{NA}}(\mathcal{Z}, \mathcal{Q}, \mathcal{L}) \geq 0$ *and* $= 0$ *if and only if* $(\mathcal{Z}, \mathcal{Q}, \mathcal{L})$ *is a product test configuration generated by some* $\eta \in N_{\mathbb{Z}}$*.*

*Proof.* By Lemma 3.15, $\xi \mapsto J(\mathcal{Z}_\xi, \mathcal{L}_\xi)$ has a unique minimizer $\xi \in N_{\mathbb{Q}}$. Assume $b \in \mathbb{N}$ satisfies $b\xi \in N_{\mathbb{Z}}$. Then we consider the test configuration $(\mathcal{Z}_\xi, \mathcal{L}_\xi)^{(b)}$ defined in (103). Then

$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{Z}, \mathcal{L}) = \mathbf{J}^{\mathrm{NA}}(\mathcal{Z}_\xi, \mathcal{L}_\xi) = b^{-1}\mathbf{J}^{\mathrm{NA}}((\mathcal{Z}_\xi, \mathcal{L}_\xi)^{(b)}) = 0. \tag{160}$$

By [14], this implies $(\mathcal{Z}_\xi, \mathcal{L}_\xi)^{(b)}$ is a product test configuration which implies $(\mathcal{Z}, \mathcal{L})$ itself is a product test configuration. □

**Proposition 3.22.** *Assume that* $(Z, Q)$ *is* $\mathbb{G}$*-uniformly Ding-stable. Then for any* $v \in \mathring{\mathrm{Val}}(X)^{\mathbb{G}}$ *with its associated filtration* $\mathcal{F}_v$*, we have:*

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_v) \geq \gamma \cdot \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) = \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{F}_v). \tag{161}$$

*Proof.* Let $(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m)$ be $m$-th approximating test configurations for $\mathcal{F}_v$ in Definition 2.23. By the $\mathbb{G}$-uniformly Ding-stability for $\mathbb{G}$, we have:

$$\mathbf{D}^{\mathrm{NA}}(\check{\mathcal{Z}}_m, \check{\mathcal{Q}}_m, \check{\mathcal{L}}_m) \geq \gamma \cdot \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\check{\mathcal{Z}}_{m,\xi}, \check{\mathcal{L}}_{m,\xi}). \tag{162}$$

Letting $m \to +\infty$ and using Proposition 2.28 and Proposition 3.16, we get the conclusion. □

**Corollary 3.23.** *If* $(Z, Q)$ *is* $\mathbb{G}$*-uniformly Ding-stable, then there exists* $\gamma' > 0$ *such that for any* $v \in \mathring{\mathrm{Val}}(Z)^{\mathbb{G}}$*,*

$$\sup_{\xi \in N_{\mathbb{R}}} \left[ A_{(Z,Q)}(v_\xi) - (1 + \gamma') \cdot S_L(v_\xi) \right] \geq 0. \tag{163}$$

*Proof.* By the paragraph above Lemma 3.21, we know that $\mathrm{Fut}_{(Z,Q)} \equiv 0$ on $\mathfrak{t}$. Because $\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_\xi) = \mathbf{D}^{\mathrm{NA}}(\mathcal{F})$, we see the inequality (161) in Proposition 3.22 can be re-written as:

$$\sup_{\xi \in N_{\mathbb{R}}} \left[ -\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) + \mathbf{L}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) - \gamma \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) \right] \geq 0. \tag{164}$$

On the other hand, recall that (69)

$$\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) = S(v_\xi). \tag{165}$$

Moreover by (70) (see [34, Proposition 2.1]), we know that:

$$\frac{1}{n}S(v_\xi) \le \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) = \mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{F}_{v_\xi}) - S(v_\xi) \le nS(v_\xi). \tag{166}$$

So, with $\gamma' = 1 + \gamma n^{-1}$, (164) implies the inequality:

$$\sup_{\xi\in N_{\mathbb{R}}} \left[\mathbf{L}^{\mathrm{NA}}(\phi_{v_\xi}) - (1+\gamma')S_L(v_\xi)\right] \ge 0,$$

where $\phi_{v_\xi} = \phi^{\mathcal{F}_{v_\xi}}$. Recall that $\phi_{v_\xi}(v_\xi) = 0$ (see Lemma 2.32). So

$$\mathbf{L}^{\mathrm{NA}}(\phi_{v_\xi}) = \inf_w (A(w) + \phi_{v_\xi}(w)) \le A(v_\xi).$$

As a consequence, we get the inequality:

$$\sup_{\xi\in N_{\mathbb{R}}} [A(v_\xi) - (1+\gamma')S_L(v_\xi)] \ge 0. \tag{167}$$

□

**Corollary 3.24.** *If $(Z, Q)$ is $\mathbb{G}$-uniformly Ding-stable, then for any $\mathbb{G}$-invariant valuation $v \in \mathring{\mathrm{Val}}(Z)$, we have $\beta(v) \ge 0$ and $\beta(v) = 0$ if and only if $v = \mathrm{wt}_\xi$ for some $\xi \in N_{\mathbb{R}}$.*

*Proof.* Fix any $v \in \mathring{\mathrm{Val}}(Z)$, if $v = \mathrm{wt}_\xi$ for some $\xi \in N_{\mathbb{R}}$, then $\beta(v) = \beta(\mathrm{wt}_\xi) = \mathrm{Fut}_{(Z,Q)}(\xi) = 0$. Otherwise, there exists $\xi \in N_{\mathbb{R}}$ such that

$$0 \le A_{(Z,Q)}(v_\xi) - (1+\gamma')S_{(Z,Q)}(v_\xi) = \beta(v_\xi) - \gamma' S_L(v_\xi), \tag{168}$$

which implies $\beta(v_\xi) \ge \gamma' S_L(v_\xi) > 0$. □

**Remark 3.25.** *We expect the converse to this result is also true.*

# 4 Proof of Theorem 1.3

*Proof.* Because $\mathrm{CM} \ge \mathbf{D}^{\mathrm{NA}}$, so (2) implies (1). (1) trivially implies (5).

We prove (1) implies (2). Take any test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L}, \eta)$ of $(X, -(K_X + D))$. Because $\mathbb{G}$ is connected linear algebraic group, we can run $\mathbb{G}$-equivariant MMP (see [3, 1.5]) as in [44] to get a special test configuration $(\mathcal{X}^s, \mathcal{L}^s)$. Moreover, there exists $d \in \mathbb{Z}_{>0}$ such that, for any $\epsilon \in [0, 1)$ and any $\xi \in N_{\mathbb{R}}$, we have:

$$d(\mathbf{D}^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \mathcal{L}) - \epsilon \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi)) \ge \mathbf{D}^{\mathrm{NA}}(\mathcal{X}^s, \mathcal{D}^s, \mathcal{L}^s) - \epsilon \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{X}^s_\xi, \mathcal{L}^s_\xi). \tag{169}$$

To verify the claim, first assume that $\xi \in N_{\mathbb{Z}}$. The calculations in [7, 33] are about variations of (differences of) intersection numbers on compactifications of test configurations under the relative MMP process studied in [44]. Recall that the compactification depends on the isomorphism between $(\mathcal{X}, \mathcal{D}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^*$ and $(X, D, L) \times \mathbb{C}^*$ (see Definition 2.16). Here we can use the compactification given by the isomorphism $\mathfrak{i}_{\eta+\xi}$ instead of $\mathfrak{i}_\eta$. Recall that (1) also implies $\mathrm{Fut}_{(X,D)} \equiv 0$ on $\mathfrak{t}$. Then (169) follows directly from the calculation in [7, 33] under the $\mathbb{G}$-equivariant MMP.

When $\xi \in N_{\mathbb{Q}}$, choose $b \in \mathbb{N}$ such that $b\xi \in N_{\mathbb{Z}}$. Then by the discussion at the end of section 3.1 the $\xi$-twisted test configuration $(\mathcal{X}_\xi, \mathcal{D}_\xi, \mathcal{L}_\xi)$ is up to base change, or rescaling in terms of non-Archimedean metric, equivalent to

$$(\mathcal{X}, \mathcal{D}, \mathcal{L})^{(b)} := (\text{ normalization of } (\mathcal{X}, \mathcal{D}, \mathcal{L}) \times_{\mathbb{C}, \mathrm{m}_d} \mathbb{C}, b\eta + b\xi) \tag{170}$$

Then we can calculate the variation of intersection numbers on $(\mathcal{X}, \mathcal{D}, \mathcal{L})^{(b)}$ to get inequality (169). For more details, see section 4.1.

By continuity, (169) holds for all $\xi \in N_{\mathbb{R}}$. Taking supremum for $\xi$ ranging from $N_{\mathbb{R}}$, we get:

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \mathcal{L}) - \epsilon \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}) \geq \mathbf{D}^{\mathrm{NA}}(\mathcal{X}^s, \mathcal{D}^s, \mathcal{L}^s) - \epsilon \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}^s, \mathcal{L}^s). \tag{171}$$

On a special test configuration, we have:

$$\mathrm{CM}(\mathcal{X}^s_\xi, \mathcal{D}^s_\xi, \mathcal{L}^s_\xi) \quad = \quad \mathbf{D}^{\mathrm{NA}}(\mathcal{X}^s_\xi, \mathcal{D}^s_\xi, \mathcal{L}^s_\xi) = \mathbf{D}^{\mathrm{NA}}(\mathcal{X}^s, \mathcal{D}^s, \mathcal{L}^s).$$

The second identity follows from (122). So we get (1) implies (2), and (5) implies (2) (and hence (1)).

Now we show (3) implies (2). For the special test configuration $(\mathcal{X}^s, \mathcal{D}^s, \mathcal{L}^s)$, if we denote $v^s = r(\mathrm{ord}(\mathcal{X}^s_0))$ where $r : \mathbb{C}(X \times \mathbb{C}) \to \mathbb{C}(X)$ the restriction map, then

$$\begin{aligned} \mathbf{D}^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \mathcal{L}) &= A_{(X,D)}(v^s) - S_L(v^s) \\ &= A_{(X,D)}(v^s_\xi) - S_L(v^s_\xi) = \mathbf{D}^{\mathrm{NA}}(\mathcal{X}^s_\xi, \mathcal{D}^s_\xi, \mathcal{L}^s_\xi). \end{aligned}$$

The first and last identities follow from the calculations in [32, 43]. In the second equality we used (130) and $\mathrm{Fut}_{(X,D)} \equiv 0$ on $\mathfrak{t}$. Moreover by (70), we have:

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{X}^s_\xi, \mathcal{L}^s_\xi) = \mathbf{J}^{\mathrm{NA}}(\mathcal{F}_{v^s_\xi}) \geq \frac{1}{n} S_L(v^s_\xi). \tag{172}$$

Hence we see that (3) implies (2).

We have pointed out in the paragraph below Definition 3.18 that $\mathbb{G}$-uniform Ding-stability implies that $\mathrm{Fut}_{(X,D)} \equiv 0$ on $\mathfrak{t}$. So (2) implying (4) follows from Corollary 3.23. Finally (4) trivially implies (3).

□

## 4.1 On the proof of inequality (169)

There are three main steps in the MMP process in [44] to obtain a special test configuration from any given test configuration. Step 1 is to use semistable reduction and run relative MMP to get the log canonical modification. Step 2 is to run MMP with rescaling to get $(\mathcal{X}^{\mathrm{ac}}, \mathcal{L}^{\mathrm{ac}})$. Step 3 is to use Fano extension to get a special test configuration $(\mathcal{X}^s, \mathcal{D}^s, \mathcal{L}^s)$. Our key to prove (169) is to adapt the calculation in [32] twisted by base change and by birational map $\bar{\sigma}_{b\xi}$ away from the central fiber. Since the intersection numbers are functorial under base change and birational morphisms, it is easy to verify the wanted inequality. We will just show the detailed calculation for the first step. The method of verification for Step 2 and Step 3 are similar as in Step 1.

**Theorem 4.1.** *Let $(X, D)$ be a log Fano pair and $(\mathcal{X}, \mathcal{D}, \mathcal{L})/\mathbb{C}$ be a normal, ample test configuration for $(X, D, -(K_X + D))$. Then there exist $d \in \mathbb{Z}_{>0}$, a projective birationally $\mathbb{C}^*$-equivariant morphism $\pi : \mathcal{X}^{\mathrm{lc}} \to \mathcal{X}^{(d)}$ and a normal, ample test configuration $(\mathcal{X}^{\mathrm{lc}}, \mathcal{D}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}})$ for $(X, -(K_X + D))$ such that*

*(1) $(\mathcal{X}^{\mathrm{lc}}, \mathcal{D}^{\mathrm{lc}} + \mathcal{X}^{\mathrm{lc}}_0)$ is log canonical.*

*(2) For any $\epsilon \in [0,1]$ and any $\xi \in N_{\mathbb{Q}}$, we have:*

$$d\left(\mathbf{D}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{D}_\xi, \mathcal{L}_\xi) - \epsilon \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi)\right) \geq \mathbf{D}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}_\xi, \mathcal{D}^{\mathrm{lc}}_\xi, \mathcal{L}^{\mathrm{lc}}_\xi) - \epsilon \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}_\xi, \mathcal{L}^{\mathrm{lc}}_\xi). \quad (173)$$

*Proof.* As in [44], there exist $d \in \mathbb{Z}_{>0}$ and the log canonical modification $\pi : \mathcal{X}^{\mathrm{lc}} \to (\mathcal{X}^{(d)}, \mathcal{X}^{(d)}_0)$. Set $\mathcal{L}^{\mathrm{lc}}_0 = \pi^* \mathrm{m}_d^* \mathcal{L}$. Let $\Delta$ be the $\mathbb{Q}$-divisor on $\mathcal{X}^{\mathrm{lc}}$ defined by

$$\mathrm{Supp}(E) \subset \mathcal{X}^{\mathrm{lc}}_0, \quad E \sim_{\mathbb{Q}} K_{\mathcal{X}^{\mathrm{lc}}/\mathbb{C}} + \mathcal{L}^{\mathrm{lc}}_0.$$

Set $\mathcal{L}^{\mathrm{lc}}_t = \mathcal{L}^{\mathrm{lc}}_0 + tE$. Then by [44, Theorem 2], $(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}})/\mathbb{C}$ is a normal, ample test configuration for $(X, -(K_X + D))$ satisfying $\mathrm{CM}(\mathcal{X}^{\mathrm{lc}}, \mathcal{D}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}}_t) \leq d \cdot \mathrm{CM}(\mathcal{X}, \mathcal{D}, \mathcal{L})$.

Let $\mathcal{X}^{\mathrm{lc}}_0 = \sum_{i=1}^p E_i$ be the irreducible decomposition and set $E := \sum_{i=1}^p e_i E_i$. Assume $e_1 \leq \cdots \leq e_p$. Then $\Delta_t := -K_{\mathcal{X}^{\mathrm{lc}}} - \mathcal{D}^{\mathrm{lc}} - \mathcal{L}^{\mathrm{lc}}_t = -(1+t)E$. Because $(\mathcal{X}^{\mathrm{lc}}, \mathcal{D}^{\mathrm{lc}} + \mathcal{X}^{\mathrm{lc}}_0)$ is log canonical,

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}, \mathcal{D}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}}_t) = \mathrm{lct}(\mathcal{X}^{\mathrm{lc}}, \Delta_t; \mathcal{X}^{\mathrm{lc}}_0) = 1 + (1+t)e_1. \quad (174)$$

Choose $b \in \mathbb{Z}_{>0}$ such that $b\xi \in N_{\mathbb{Z}}$. We consider the following commutative diagrams, where $\mathcal{Z}$ is the normalization of the graph $\bar{\sigma}_{b\xi} \circ \mathrm{i}_{b\eta}$.

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & \swarrow^{\Pi} & & \searrow^{\Theta} & \\ (X \times \mathbb{P}^1)^{(b)} & \overset{\mathrm{i}_{b\eta}}{\dashrightarrow} & (\mathcal{X}^{\mathrm{lc}})^{(b)} & \overset{\bar{\sigma}_{b\xi}}{\dashrightarrow} & (\mathcal{X}^{\mathrm{lc}})^{(b)} \\ \downarrow \mathrm{m}_b & & \downarrow \mathrm{m}_b & & \downarrow \mathrm{m}_b \\ X \times \mathbb{P}^1 & \overset{\mathrm{i}_\eta}{\dashrightarrow} & \mathcal{X}^{\mathrm{lc}} & & \mathcal{X}^{\mathrm{lc}} \end{array} \quad (175)$$

Set $\tilde{\phi}_{t,b\xi} := \Theta^* \mathrm{m}_b^* \bar{\mathcal{L}}^{\mathrm{lc}}_t$ and $\tilde{\psi} := \Pi^* \mathrm{m}_b^* p_1^*(-(K_X + D))$. Note that $\mathbf{D}^{\mathrm{NA}}$ and $\mathbf{L}^{\mathrm{NA}}$ are multiplicative under base change (see [32, Proposition 2.5.(3)]). Moreover, $\mathbf{L}^{\mathrm{NA}}$ is invariant under twisting: $\mathbf{L}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}_\xi, \mathcal{D}^{\mathrm{lc}}_\xi, \mathcal{L}^{\mathrm{lc}}_{t,\xi}) = \mathbf{L}(\mathcal{X}^{\mathrm{lc}}, \mathcal{D}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}}_t)$ (by (99)). Then we have:

$$\begin{aligned}
& (n+1)V \left[d\left(\mathbf{D}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{D}_\xi, \mathcal{L}_\xi) - \epsilon \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi)\right)\right. \\
& \qquad \left. - \left(\mathbf{D}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}_\xi, \mathcal{D}^{\mathrm{lc}}_\xi, \mathcal{L}^{\mathrm{lc}}_\xi) - \epsilon \mathbf{J}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}_\xi, \mathcal{L}^{\mathrm{lc}}_\xi)\right)\right] \\
= \; & (n+1)Vb^{-1}\left[\left(\mathbf{D}^{\mathrm{NA}}\left((\mathcal{X}_\xi, \mathcal{D}_\xi, \mathcal{L}_{0,\xi})^{(b)}\right) - \epsilon \mathbf{J}^{\mathrm{NA}}\left((\mathcal{X}_\xi, \mathcal{L}_{0,\xi})^{(b)}\right)\right)\right. \\
& \qquad \left. - \left(\mathbf{D}^{\mathrm{NA}}\left((\mathcal{X}^{\mathrm{lc}}_\xi, \mathcal{D}^{\mathrm{lc}}_\xi, \mathcal{L}^{\mathrm{lc}}_\xi)^{(b)}\right) - \epsilon \mathbf{J}^{\mathrm{NA}}\left((\mathcal{X}^{\mathrm{lc}}_\xi, \mathcal{L}^{\mathrm{lc}}_\xi)^{(b)}\right)\right)\right] \\
= \; & b^{-1}(1-\epsilon)\left(\tilde{\phi}^{\cdot n+1}_{t,b\xi} - \tilde{\phi}^{\cdot n+1}_{0,b\xi}\right) + b^{-1}\epsilon(n+1)t(\tilde{\psi}^{\cdot n} \cdot \Theta^* \mathrm{m}_b^* E) - (n+1)te_1 V \\
= \; & (1-\epsilon)t\left(b^{-1}(\tilde{\phi}^{\cdot n+1}_{t,b\xi} - \tilde{\phi}^{\cdot n+1}_{0,b\xi}) - (n+1)e_1 V\right) + \epsilon(n+1)t\left(b^{-1}\tilde{\psi}^{\cdot n}\Theta^* \mathrm{m}_b^* E - e_1 V\right) \\
= \; & (1-\epsilon)tb^{-1}\sum_{i=0}^{n}\left(\tilde{\phi}^{\cdot i}_{t,b\xi} \cdot \tilde{\phi}^{n-i}_{0,b\xi} \cdot \Theta^* \mathrm{m}_b^* \sum_{j=1}^{p}(e_j - e_1)E_j\right) \\
& \qquad + b^{-1}\epsilon(n+1)t\left(\tilde{\psi}^{\cdot n} \cdot \Theta^* \mathrm{m}_b^* \sum_{j=1}^{p}(e_j - e_1)E_j\right) \geq 0.
\end{aligned}$$

□

### 4.2 An alternative proof of the valuative criterion for $\mathbb{G}$-uniform Ding stability

Here we provide a proof of the valuative criterion for $\mathbb{G}$-uniform Ding-stability without using the MMP program. In other words, we prove the equivalence of (2) $\Rightarrow$ (3) in Theorem 1.3. Since (2) implies (3) by Corollary 3.23, we just need to show the other direction. Our argument is motivated by Boucksom-Jonsson's work in [17] and will also be used in the proof of existence result in section 5.4. We first claim that it suffices to prove the following inequality: for any non-Archimedean metric $\phi = \phi_{(\mathcal{X},\mathcal{L})}$ coming from $\mathbb{G}$-equivariant semi-ample test configuration,

$$\inf_{v\in (X^{\rm div}_{\mathbb{Q}})^{\mathbb{G}}}(S_L(v)+\phi(v)) \geq \inf_{v\in X^{\rm div}}(S_L(v)+\phi(v)) \geq \mathbf{E}^{\rm NA}(\phi). \tag{176}$$

Assume that this is true. By the expression of $\mathbf{L}^{\rm NA}$ in (91), we can find $v_k \in (X^{\rm div}_{\mathbb{Q}})^{\mathbb{G}}$ such that

$$\mathbf{L}^{\rm NA}(\phi) \leq A_X(v_k)+\phi(v_k) \leq \mathbf{L}^{\rm NA}(\phi)+\frac{1}{k}. \tag{177}$$

Assuming the valuative condition, there exists $\xi_k \in N_{\mathbb{R}}$ such that $A(v_{k,-\xi_k}) \geq \delta S_L(v_{k,-\xi_k})$. By density, we can assume $\xi_k \in N_{\mathbb{Q}}$ so that up to base change, $\phi_{\xi_k}$ is equivalent to a semi-ample test configuration. So we can apply inequality (176) to $\phi_{\xi_k}$ to get:

$$\begin{aligned} A(v_k)+\phi(v_k) &= A(v_{k,-\xi_k})+\phi_{\xi_k}(v_{k,-\xi_k}) \geq \delta S_L(v_{k,-\xi_k})+\phi_{\xi_k}(v_{k,-\xi_k}) \\ &\geq \delta \mathbf{E}^{\rm NA}(\delta^{-1}\phi_{\xi_k}). \end{aligned} \tag{178}$$

The first equality uses the identity (123). Combining (177)-(178), we get:

$$\begin{aligned} \mathbf{D}^{\rm NA}(\phi) &= -\mathbf{E}^{\rm NA}(\phi)+\mathbf{L}^{\rm NA}(\phi) \geq -\mathbf{E}^{\rm NA}(\phi)+A(v_k)+\phi(v_k)-\frac{1}{k} \\ &\geq -\mathbf{E}^{\rm NA}(\phi_{\xi_k})+\delta\mathbf{E}^{\rm NA}(\delta^{-1}\phi_{\xi_k})-\frac{1}{k} = \delta\mathbf{J}^{\rm NA}(\delta^{-1}\phi_{\xi_k})-\mathbf{J}^{\rm NA}(\phi_{\xi_k})-\frac{1}{k} \\ &\geq (1-\delta^{-1/n})\mathbf{J}^{\rm NA}(\phi_{\xi_k})-\frac{1}{k} \geq (1-\delta^{-1/n})\mathbf{J}^{\rm NA}_{\mathbb{T}}(\phi)-\frac{1}{k}, \end{aligned}$$

where we used the non-Archimedean version of Ding's inequality ([16, Lemma 6.17]).

Coming back to the proof of the inequality (176), we give a different proof with that in [17] (without using the Legendre duality by viewing $S_L(v)$ as $\mathbf{E}^*(\delta_v)$). To do this, we use the explicit description of the filtration $\mathcal{F} = \mathcal{F}_{(\mathcal{X},\mathcal{L})}$ associated to a normal semi-ample test configuration in (65) and compare it with the filtration $\mathcal{F}_v$ induced by any divisorial valuation $v$. Using similar notation as there, we set $\mathcal{L} = \rho^* L_{\mathbb{C}}+D$ with $D = \sum_E a_E E$ where $E$ runs over irreducible components of the central fibre $X_0 = \sum_E b_E E$. By (83), we know that, for any fixed divisorial valuation $v$ over $X$:

$$\phi(v) = \phi_{(\mathcal{X},\mathcal{L})}(v) = G(v)(D) = \sum_E a_E G(v)(E) =: a. \tag{179}$$

Now for any $s \in \mathcal{F}^x R_m$, $r(\mathrm{ord}_E)(s)+m\ell_0\cdot \mathrm{ord}_E(D) \geq x b_E$ by (65). This implies that:

$$\begin{aligned} v(s) &= G(v)(\bar{s}) = \sum_E G(v)(E)\mathrm{ord}_E(\bar{s}) \geq \sum_E G(v)(E)(x b_E - m\ell_0 a_E) \\ &= xG(v)(t)-m\ell_0\sum_E a_E G(v)(E) = x-m\ell_0 a. \end{aligned}$$

So we get $\mathcal{F}^x R_m \subseteq \mathcal{F}^{x-m\ell_0 a}_v R_m$. As a consequence, $\mathrm{vol}(\mathcal{F}^{(t)}) \leq \mathrm{vol}(\mathcal{F}_v^{(t-\ell_0 a)})$. Because $\lambda_{\min} = \inf\{t\in\mathbb{R}; \mathrm{vol}(\mathcal{F}^{(t)}) < \ell_0^n V\}$ by [14, Corollary 5.4] and $\mathrm{vol}(\mathcal{F}_v^{(t)}) < V\ell_0^n$ when $t>0$

(by Izumi's inequality, see [43, 5]), we easily get the inequality $\lambda_{\min} \le \ell_0 a$. We can then calculate as follows to get the wanted inequality:

$$\begin{aligned}
\mathbf{E}^{\mathrm{NA}}(\phi) &= -\frac{1}{V\ell_0^n}\int_{\mathbb{R}} \frac{x}{\ell_0} d\mathrm{vol}(\mathcal{F}^{(x)}) = \frac{\lambda_{\min}}{\ell_0} + \frac{1}{V\ell_0^{n+1}}\int_{\lambda_{\min}}^{+\infty} \mathrm{vol}(\mathcal{F}^{(x)})dx \\
&\le a + \frac{1}{V\ell_0^{n+1}}\int_{\ell_0 a}^{+\infty} \mathrm{vol}(\mathcal{F}^{(x)})dx \le a + \frac{1}{V\ell_0^{n+1}}\int_{\ell_0 a}^{+\infty} \mathrm{vol}(\mathcal{F}_v^{(x-\ell_0 a)})dx \\
&= a + \frac{1}{V\ell_0^{n+1}}\int_0^{+\infty} \mathrm{vol}(\mathcal{F}_v^{(t)})dt = \phi(v) + S_L(v).
\end{aligned}$$

The second identity is obtained by integration by parts (which holds even if $d\mathrm{vol}(\mathcal{F}^{(x)})$ has a Dirac mass at $\lambda_{\max}(\mathcal{F})$). The second inequality is because the function $y \mapsto y + \frac{1}{V\ell_0^{n+1}}\int_y^\infty \mathrm{vol}(\mathcal{F}^{(x)})dx$ is an increasing function of $y \in \mathbb{R}$ (which is constant for $y \le \lambda_{\min}(\mathcal{F})$). The last identity uses (179) and (68).

# 5 Proof of Theorem 1.2 and Theorem 1.5

The necessary part of Theorem 1.5 immediately follows from Theorem 2.15 and Theorem 3.14. So the rest of this paper is devoted to proving Theorem 1.2.

By Theorem 2.14, we just need to prove the Mabuchi energy is $\mathbb{G}$-proper. The general strategy is of course motivated by [8] and our previous work [46]. However due to the various complications caused by twists, we need to re-work out the argument more carefully. One main point is that we only work with $\mathbb{K}$-invariant (in particular $(S^1)^r$-invariant) metrics. The proof is processed by contradiction argument. So we assume that the Mabuchi energy is not $\mathbb{G}$-proper.

## 5.1 Step 1: Construct a destabilizing geodesic ray

In this step, assuming that the Mabuchi energy $\mathbf{M} = \mathbf{M}_{(X,D)}$, is not $\mathbb{G}$-proper, we will find a destabilizing geodesic ray $\Phi = (\varphi(s))$ in $\mathcal{E}^1(X, L)^{\mathbb{K}}$ such that

(1) The Ding energy is decreasing along $\Phi = \{\varphi(s)\}$ for any $\xi \in N_{\mathbb{R}}$:

$$\mathbf{D}'^{\infty}(\Phi) = \lim_{s\to+\infty} \frac{\mathbf{D}(\varphi(s))}{2s} \le 0. \tag{180}$$

(2) we have the normalization:

$$\sup(\varphi(s) - \psi_0) = 0, \quad \mathbf{E}_{\psi_0}(\varphi(s)) = -s. \tag{181}$$

(3) For any $\xi \in N_{\mathbb{R}}$, the geodesic $\Phi_\xi := \{\varphi_\xi(s)\} := \{\sigma_\xi(s)^*\varphi(s)\}$ satisfies:

$$\mathbf{J}'^{\infty}(\Phi_\xi) = \lim_{s\to+\infty} \frac{\mathbf{J}_\psi(\sigma_\xi(s)^*\varphi(s))}{2s} > 0. \tag{182}$$

The argument for constructing such a destabilising geodesic ray is similar to the arguments in [7, 8]. All energy functionals in this step are on $X$ itself as defined in (19)-(30). Assume the Mabuchi energy $\mathbf{M} = \mathbf{M}_{\psi_0}$ (see (30)) is not $\mathbb{G}$-proper. Then choosing $\gamma_j \to 0$, we can pick a sequence $\{u_j\}_{j=1}^\infty \in (\mathcal{E}^1)^{\mathbb{K}} = (\mathcal{E}^1(X,\omega))^{\mathbb{K}}$ as in [46, 4.1] such that $\varphi_j = \psi_0 + u_j$ satisfies:

$$\mathbf{D}(\varphi_j) \le \mathbf{M}(\varphi_j) \le \gamma_j \mathbf{J}_{\mathbb{T}}(\varphi_j) - j \le \gamma_j \mathbf{J}(\sigma^*\varphi_j) - j \tag{183}$$

for any $\sigma \in \mathbb{T}$. Because of Lemma 2.12, we can assume that:

$$\mathbf{J}_{\mathbb{T}}(\varphi_j) = \inf_{\sigma \in \mathbb{T}} \mathbf{J}(\sigma^* \varphi_j). \tag{184}$$

We normalize $\varphi_j$ such that $\sup(\varphi_j - \psi_0) = 0$. The inequality

$$\mathbf{M}(\varphi_j) = \mathbf{M}(\sigma^* \varphi_j) \geq C - n\mathbf{J}(\sigma^* \varphi_j)$$

implies that for any $\sigma \in \mathbb{T}$,

$$\mathbf{J}(\sigma^* \varphi_j) \geq \frac{j + C}{n + \gamma_j} \to +\infty \quad \text{as } j \to +\infty,$$

and hence $\mathbf{E}(\varphi_j) \leq -\mathbf{J}(\varphi_j) \to -\infty$.

Denote $V = (2\pi)^n (-K_X - D)^{\cdot n}$. By the work [24, 28], we can connect $\psi_0$ and $\varphi_j$ by a unit speed geodesic segment $\{\varphi_j(s)\} \in \mathrm{PSH}_{\mathrm{bd}}(X, L)^{\mathbb{K}}$ parametrized so that $S_j := -\mathbf{E}(\varphi_j) \to +\infty$ with $s \in [0, S_j]$. In particular, $\mathbf{E}(\varphi_j(s)) = -s$. Then $\psi_0$ and $\varphi_{j,\xi} := \sigma_\xi(S_j)^* \varphi_j$ is connected by the geodesic segment $\sigma_\xi(s)^* \varphi_j$, $s \in [0, S_j]$.

By [46, 4.1.2] (see also [5, 9]), $\mathbf{M}$ is convex along geodesic segment. So we get,

$$\begin{aligned}
\mathbf{D}(\varphi_j(s)) &\leq \mathbf{M}(\varphi_j(s)) \leq \frac{S_j - s}{S_j} \mathbf{M}(\psi_0) + \frac{s}{S_j} \mathbf{M}(\varphi_j) \\
&\leq C + \frac{s}{S_j}(\gamma_j \mathbf{J}(\varphi_j) - j) \leq C + \frac{s}{S_j} \gamma_j \mathbf{J}(\varphi_j).
\end{aligned} \tag{185}$$

Using $\mathbf{M} \geq \mathbf{H} - n\mathbf{J}$, we get $\mathbf{H}(\varphi_j(s)) \leq (\gamma_j + n)s + C$. So for any fixed $S > 0$ and $s \leq S$, the metrics $\varphi_j(s)$ lie in the set:

$$\mathcal{K}_S := \{\varphi \in \mathcal{E}^1; \sup(\varphi - \psi_0) = 0 \text{ and } \mathbf{H}(\varphi) \leq (\gamma_j + n)S + C\}.$$

This is a compact subset of the metric space $(\mathcal{E}^1, d_1)$ by Theorem 2.11 from [6]. So, by arguing as in [7], after passing to a subsequence, $\{\varphi_j(s)\}$ converges to a geodesic ray $\Phi := \{\varphi(s)\}_{s \geq 0}$ in $(\mathcal{E}^1)^{\mathbb{K}}$, uniformly for each compact time interval. Moreover $\{\varphi(s)\}_{s \in \mathbb{R}}$ satisfies

$$\lim_{s \to +\infty} \frac{\mathbf{D}(\varphi(s))}{2s} \leq 0, \quad \sup(\varphi(s) - \psi_0) = 0, \quad \mathbf{E}(\varphi(s)) = -s. \tag{186}$$

For any $\xi \in N_{\mathbb{R}}$, by (184) we have

$$\mathbf{J}(\sigma_\xi(S_j)^* \varphi_j) \geq \mathbf{J}(\varphi_j) = -\mathbf{E}(\varphi_j) + O(1) = S_j + O(1) \to +\infty. \tag{187}$$

The second identity uses Lemma 2.1. Moreover $\{\sigma_\xi(s)^* \varphi_j(s)\}_{s \in [0, S_j]}$ converges strongly to the geodesic ray $\Phi_\xi := \{\sigma_\xi(s)^* \varphi(s)\}_{s \geq 0}$. So we get, for any $\xi \in N_{\mathbb{R}}$,

$$\lim_{s \to +\infty} \mathbf{J}_\psi(\sigma_\xi(s)^* \varphi(s)) = +\infty \tag{188}$$

This implies that $\{\sigma_\xi(s)^* \varphi(s)\}$ is a nontrivial geodesic, because (for $\mathbf{E}$-normalized potentials) $\mathbf{J}$-energy is comparable to $d_1$-distance which is linear along geodesics (see [24, (31)], [28, Theorem 3.6]). In particular, for any $\xi \in N_{\mathbb{R}}$

$$\mathbf{J}'^{\infty}(\Phi_\xi) := \lim_{s \to +\infty} \frac{\mathbf{J}_\psi(\sigma_\xi(s)^* \varphi(s))}{2s} > 0. \tag{189}$$

**Proposition 5.1** (see [39, Proposition 1.6]). *Let $\Phi = \{\varphi(s)\}_{s \in \mathbb{R}} \subset \mathcal{E}^1(L)^{(S^1)^r}$ be a geodesic ray. The function Let $(s, \xi) \to \mathbf{J}(\sigma_\xi(s)^* \varphi(s))$ is convex in $(s, \xi) \in \mathbb{R} \times N_{\mathbb{R}}$.*

*Proof.* Choose any $\xi_0, \xi' \in N_{\mathbb{R}}$. Consider the holomorphic map (see (32)):

$$F : X \times \mathbb{C} \times \mathbb{C} \to X \times \mathbb{C}, \quad (x, \mathfrak{s} = s + iu, \mathfrak{c} = c + id) \mapsto (\sigma_{\xi_0}(\mathfrak{s})\sigma_{\xi'}(\mathfrak{c}\mathfrak{s}) \cdot x, \mathfrak{s}). \tag{190}$$

Then $F^*\Phi$ is a positively curved finite energy Hermitian metric on $p_1^*L$ where $p_1 : X \times \mathbb{C} \times \mathbb{C} \to X$ is the projection. For any $c \in \mathbb{R}$, denote $\xi_c := \xi_0 + c\xi'$.

Note that, because $\exp(J\xi), \exp(J\xi') \in (S^1)^r$ and $\varphi(s) \in \mathcal{E}^1(L)^{(S^1)^r}$, we have:

$$\begin{aligned} F^*\Phi &= (\exp(s\xi_0)\exp(uJ\xi_0))^* \exp((sc - ud)\xi')^* \exp((sd + uc)J\xi')^*\varphi(s) \\ &= \exp(s\xi_0)^* \exp((sc - ud)\xi')^*\varphi(s). \end{aligned}$$

In particular, $F^*\Phi|_{u=0}$ is the twisted geodesic ray $\sigma_{\xi_0+c\xi'}(s)^*\varphi(s)$. Because $F$ is holomorphic we know that $\sqrt{-1}\partial\bar{\partial}F^*\Phi \geq 0$. Moreover, by the integration along the fibre formula, we have:

$$\sqrt{-1}\partial\bar{\partial}_{(\mathfrak{s},\mathfrak{c})} J(\sigma_\xi(s)^*\varphi(s)) = \frac{1}{(2\pi)^n L^{\cdot n}} \int_X (\sqrt{-1}\partial\bar{\partial}F^*\Phi) \wedge (\sqrt{-1}\partial\bar{\partial}\psi)^n \geq 0. \tag{191}$$

As a consequence $f(s, c) := J(\sigma_{\xi_0+c\xi'}(s)^*\varphi(s))$ is convex.

□

**Proposition 5.2.** *The function $\xi \mapsto \mathbf{J}'^{\infty}(\Phi_\xi)$ is convex in $\xi \in N_{\mathbb{R}}$.*

*Proof.* Using the notations in the proof of the Proposition 5.1, we consider the convex function $f(s, c) := J(\sigma_{\xi_0+c\xi'}(s)^*\varphi(s))$. Then for any $0 < c_1 < c_2$, by convexity we have

$$f(s, c_1) \leq (1 - \frac{c_1}{c_2})f(s, 0) + \frac{c_1}{c_2}f(s, c_2). \tag{192}$$

Dividing both sides by $s$ and letting $s \to +\infty$, we get the wanted convexity:

$$\mathbf{J}'^{\infty}(\Phi_{\xi_0+c_1\xi'}) \leq (1 - \frac{c_1}{c_2})\mathbf{J}'^{\infty}(\Phi_{\xi_0}) + \frac{c_1}{c_2}\mathbf{J}'^{\infty}(\Phi_{\xi+c_2\xi'}). \tag{193}$$

□

Because a convex function on $N_{\mathbb{R}} \cong \mathbb{R}^r$ is continuous, it obtains a minimum on compact set. Combing this with (189) we get:

**Corollary 5.3.** *For any $C > 0$ there exists $\chi = \chi(C, \Phi) > 0$ such that for any $\xi$ satisfying $|\xi| < C$, $\mathbf{J}'^{\infty}(\Phi_\xi) \geq \chi > 0$.*

**Remark 5.4.** *We expect that the function $\xi \mapsto \mathbf{J}'^{\infty}(\Phi_\xi)$ is proper on $N_{\mathbb{R}}$. This is would be a generalization of Lemma 3.15 and in our case together with Corollary 5.3 would imply that:*

$$\inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}'^{\infty}(\Phi_\xi) > 0. \tag{194}$$

*Note that by the vanishing of Futaki invariant we have*

$$\mathbf{J}(\sigma_\xi(s)^*\varphi(s)) = \mathbf{\Lambda}(\sigma_\xi(s)^*\varphi) - \mathbf{E}(\sigma_\xi(s)^*\varphi(s)) = \mathbf{\Lambda}(\sigma_\xi(s)^*\varphi(s)) - \mathbf{E}(\varphi(s)),$$

*Moreover we can decompose:*

$$\begin{aligned} \mathbf{J}(\sigma_\xi(s)^*\varphi(s)) &= \frac{1}{(2\pi)^n L^{\cdot n}} \int_X (\sigma_\xi^*(\varphi) - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^n \\ &= \frac{1}{(2\pi)^n L^{\cdot n}} \int_X \sigma_\xi^*(\varphi - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^n + \frac{1}{(2\pi)^n L^{\cdot n}} \int_X (\sigma_\xi^*\psi - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^n \\ &=: f(\xi, s) + \mathbf{\Lambda}(\sigma_\xi^*\psi) \end{aligned}$$

*where* $f(\xi, s) = \frac{1}{(2\pi)^n L^{\cdot n}} \int_X \sigma_\xi(s)^*(\varphi(s) - \psi)(\sqrt{-1}\partial\bar{\partial}\psi)^n$. *From this decomposition we see that:*

$$\mathbf{J}'^{\infty}(\Phi_\xi) = \lim_{s \to +\infty} \frac{f(\xi, s)}{2s} + \mathbf{\Lambda}^{\mathrm{NA}}(\xi) =: g(\xi) + \mathbf{\Lambda}^{\mathrm{NA}}(\xi). \tag{195}$$

*From the proof of Lemma 3.15, we know that* $\xi \mapsto \mathbf{\Lambda}^{\mathrm{NA}}(\xi)$ *is proper on* $N_{\mathbb{R}}$. *So it is sufficient to show that* $\xi \mapsto g(\xi)$ *is bounded from below. This is true if we know that* $X$ *is smooth and the geodesic ray* $\Phi$ *is* $C^1$ *or locally bounded. Indeed, in this case* $f(\xi, x) \geq \inf(\varphi - \psi)$ *and by [23, Theorem 1], we know that* $\inf(\varphi - \psi)$ *is linear with respect to* $s$. *Moreover Darvas-Lu in their recent preprint [25] showed (by rather involved arguments) that, at least when* $X$ *is smooth, one can indeed assume that* $\Phi$ *is* $C^{1,1}$.

## 5.2 Step 2: Perturbed and twisted test configurations

Fix a $\mathbb{G}$-equivariant resolution of singularities $\mu : Y \to X$ such that $\mu$ is an isomorphism over $X^{\mathrm{reg}}$, $\mu^{-1}(X^{\mathrm{sing}}) = \sum_{k=1}^{g} E_k$ is a $\mathbb{G}$-invariant simple normal crossing divisor and that there exist $\theta_k \in \mathbb{Q}_{>0}$ for $k = 1, \dots, g$ such that $E_\theta = \sum_{k=1}^{g} \theta_k E_k$ satisfies $P := P_\theta = \mu^* L - E_\theta$ is an ample $\mathbb{Q}$-divisor over $Y$. We can then choose and fix a smooth $\mathbb{K}$-invariant Hermitian metric $\varphi_P$ on $P$ such that $\sqrt{-1}\partial\bar{\partial}\varphi_P > 0$.

For any $\epsilon \in \mathbb{Q}_{>0}$, define line bundles on $Y$ by

$$\hat{L}_\epsilon := (1+\epsilon)\mu^* L - \epsilon E_\theta = \mu^* L + \epsilon P, \quad L_\epsilon = \frac{1}{1+\epsilon}\hat{L}_\epsilon. \tag{196}$$

Then $\hat{L}_\epsilon$ is a positive $\mathbb{Q}$-line bundle on $Y$. Define a smooth reference metric on $\hat{L}_\epsilon$ by $\hat{\psi}_\epsilon = \psi_0 + \epsilon\varphi_P \in (\mathcal{E}^1(X, L_\epsilon))^{\mathbb{K}}$. Let $\Phi = \{\varphi(s)\}$ be a geodesic ray in $(\mathcal{E}^1(X, L))^{\mathbb{K}}$ constructed in the above subsection, which satisfies:

$$\sup_X(\varphi(s) - \psi_0) = 0, \quad \mathbf{E}_{\psi_0}(\varphi(s)) = -s. \tag{197}$$

.

In this section we will first construct a sequence of test configurations of $(Y, \hat{L}_\epsilon)$ using the method from [7]. Denote by $p_i', i = 1, 2$ the projection of $Y \times \mathbb{C}$ to the two factors. Define a singular and a smooth $\mathbb{K}$-invariant Hermitian metric on $p_1'^* \hat{L}_\epsilon$ by

$$\hat{\Phi}_\epsilon := (\mu \times \mathrm{id})^*(\Phi) + \epsilon\, p_1'^*(\varphi_P), \quad \hat{\Psi}_\epsilon := p_1'^*(\mu^*\psi_0 + \epsilon\, \varphi_P). \tag{198}$$

Then $\sqrt{-1}\partial\bar{\partial}\hat{\Phi}_\epsilon \geq 0$, $\sqrt{-1}\partial\bar{\partial}\hat{\Psi}_\epsilon \geq 0$. Fix a very ample line bundle $H'$ over $Y$. Consider the following coherent sheaf:

$$\begin{aligned} \mathcal{F}_{\epsilon,m} &:= \mathcal{O}_Y(p_1'^*(m\hat{L}_\epsilon) \otimes \mathcal{J}(Y, m\hat{\Phi}_\epsilon)) \\ &= \mathcal{O}(K_Y + m\mu^* L + (m\epsilon P - K_Y - (n+1)H') + (n+1)H') \otimes \mathcal{J}(Y; m\mu^*\Phi)). \end{aligned}$$

Because $P$ is positive, for $m \gg \epsilon^{-1}$ and sufficiently divisible, $m\epsilon P - K_Y - (n+1)H'$ is an ample line bundle on $Y$. In this case, by Nadel vanishing theorem, for any $j \geq 1$,

$$R^j(p_2')_*(\mathcal{F}_{\epsilon,m} \otimes p_1^* H'^{-j}) = 0.$$

By the relative Castelnuovo-Mumford criterion, $\mathcal{F}_{\epsilon,m}$ is $p_2'$-globally generated. Because $\mathbb{D}$ is Stein, $\mathcal{O}(p_1'^*(m\hat{L}_\epsilon) \otimes \mathcal{J}(m\hat{\Phi}_\epsilon))$ is generated by global sections on $Y \times \mathbb{D}$ if $m \gg \epsilon^{-1}$ and $m$ is sufficiently divisible.

Let $\pi_m' : \mathcal{Y}_{\epsilon,m} \to Y_{\mathbb{C}}$ denote the normalized blow-up of $Y \times \mathbb{C}$ along $\mathcal{J}(m\hat{\Phi}_\epsilon)$, with exceptional divisor $E_{\epsilon,m}$ and set

$$\hat{\mathcal{L}}_{\epsilon,m} := \pi_m'^* p_1'^* \hat{L}_\epsilon - \frac{1}{m} E_{\epsilon,m}, \quad \mathcal{L}_{\epsilon,m} = \frac{1}{1+\epsilon}\hat{\mathcal{L}}_{\epsilon,m}. \tag{199}$$

Then $(\mathcal{Y}_{\epsilon,m}, \hat{\mathcal{L}}_{\epsilon,m})$ is a $\mathbb{G}$-equivariant normal semi-ample test configuration for $(Y, \hat{L}_\epsilon)$. To see $\mathbb{G}$-equivariance, note that by the $\mathbb{K}$-invariance of $\hat{\Phi}_\epsilon$, $\mathcal{J}(Y, m\hat{\Phi})$ is invariant under the action of $\mathbb{K}$ on $\mathcal{O}_Y$. Because $\mathbb{G} = \mathbb{K}^{\mathbb{C}}$, the invariance under $\mathbb{G}$ action follows from the biholomorphicity of the $\mathbb{G}$-action.

The associated non-Archimedean metric $\hat{\phi}_{\epsilon,m} \in \mathcal{H}^{\rm NA}(\hat{L}_\epsilon)$ is given by:

$$\hat{\phi}_{\epsilon,m}(w) = -\frac{1}{m} G(w)(\mathcal{J}(m\Phi)), \tag{200}$$

for each $w \in \mathring{\rm Val}(Y)$. Note that we used the fact that, since $\varphi_P$ is a smooth Hermitian metric,

$$\mathcal{J}(m\hat{\Phi}_\epsilon) = \mathcal{J}(m\mu^*\Phi) =: \mathcal{J}(m\Phi). \tag{201}$$

We will denote by $\hat{\Phi}_{\epsilon,m} = \{\hat{\varphi}_{\epsilon,m}(s)\}$ the geodesic ray associated to $(\mathcal{Y}_{\epsilon,m}, \hat{\mathcal{L}}_{\epsilon,m})$. By Demailly's regularization result ([27, Proposition 3.1]), $\hat{\Phi}_{\epsilon,m}$ is less singular then $\hat{\Phi}_\epsilon$. As a consequence, $\hat{\Phi}_{\epsilon,m,\xi} := \{\sigma_\xi(s)^*\varphi_{\epsilon,m}(s)\}_{s\in[0,+\infty)}$ is less singular than $\hat{\Phi}_{\epsilon,\xi} = \{\sigma_\xi(s)^*\varphi_\epsilon(s)\}_{s\in[0,+\infty)}$. By the monotonicity of $\mathbf{E}$ and $\mathbf{\Lambda}$ energy (see (23)), we get:

$$\begin{aligned} \mathbf{E}^{\rm NA}_{\hat{L}_\epsilon}(\hat{\phi}_{\epsilon,m,\xi}) &= \lim_{s\to+\infty} \frac{\mathbf{E}_{\hat{\psi}_\epsilon}(\sigma_\xi(s)^*\hat{\varphi}_{\epsilon,m}(s))}{2s} \\ &\ge \lim_{s\to+\infty} \frac{\mathbf{E}_{\hat{\psi}_\epsilon}(\sigma_\xi(s)^*\hat{\varphi}_\epsilon(s))}{2s} =: \mathbf{E}'^{\infty}_{\hat{\psi}_\epsilon}(\hat{\Phi}_{\epsilon,\xi}). \end{aligned} \tag{202}$$

$$\begin{aligned} \mathbf{\Lambda}^{\rm NA}_{\hat{L}_\epsilon}(\hat{\phi}_{\epsilon,m,\xi}) &= \lim_{s\to+\infty} \frac{\mathbf{\Lambda}_{\hat{\psi}_\epsilon}(\sigma_\xi(s)^*\hat{\varphi}_{\epsilon,m})}{2s} \\ &\ge \lim_{s\to+\infty} \frac{\mathbf{\Lambda}_{\hat{\psi}_\epsilon}(\sigma_\xi(s)^*\hat{\varphi}_\epsilon(s))}{2s} =: \mathbf{\Lambda}'^{\infty}_{\hat{\psi}_\epsilon}(\hat{\Phi}_{\epsilon,\xi}). \end{aligned} \tag{203}$$

The following convergence will be important for us.

**Lemma 5.5.** *With the above notations and assuming that $\Phi = \{\varphi(s)\}$ satisfies (197), for any $\xi \in N_{\mathbb{R}}$ the following identities hold true:*

$$\lim_{\epsilon\to0} \mathbf{E}'^{\infty}_{\hat{\psi}_\epsilon}(\hat{\Phi}_{\epsilon,\xi}) = \lim_{s\to+\infty} \frac{\mathbf{E}_\psi(\varphi_\xi(s))}{2s} =: \mathbf{E}'^{\infty}(\Phi_\xi). \tag{204}$$

$$\lim_{\epsilon\to0} \mathbf{\Lambda}'^{\infty}_{\hat{\psi}_\epsilon}(\hat{\Phi}_{\epsilon,\xi}) = \lim_{s\to+\infty} \frac{\mathbf{\Lambda}_\psi(\varphi_\xi(s))}{2s} =: \mathbf{\Lambda}'^{\infty}(\Phi_\xi). \tag{205}$$

*Proof.* Because $\mathbf{E}$ satisfies cocycle condition and is affine along geodesics, it is easy to verify that, for any $\varphi \in \mathcal{E}^1(\hat{L}_\epsilon)$,

$$\begin{aligned} \mathbf{E}_{\hat{\psi}_\epsilon}(\sigma_\xi(s)^*\hat{\varphi}_\epsilon) &= \mathbf{E}_{\sigma_\xi(s)^*\hat{\psi}_\epsilon}(\sigma_\xi(s)^*\varphi) + E_{\hat{\psi}_\epsilon}(\sigma_\xi(s)^*\hat{\psi}_\epsilon) \\ &= \mathbf{E}_{\hat{\psi}_\epsilon}(\hat{\varphi}_\epsilon) + {\rm CW}_{\hat{L}_\epsilon}(\xi)\cdot s, \end{aligned}$$

where ${\rm CW}_{\hat{L}_\epsilon} = {\rm CW}_L + \epsilon\cdot{\rm CW}_P$ is the Chow weight of $\xi$ (see (72)). It was proved in [46] that:

$$\lim_{\epsilon\to0} \mathbf{E}'^{\infty}(\hat{\Phi}_\epsilon) = \mathbf{E}'^{\infty}(\Phi). \tag{206}$$

These combine to give (204). Next we prove (205). By the definition of $\mathbf{\Lambda}$-energy (see (20))

$$\begin{aligned}(2\pi)^n \hat{L}_\epsilon^{\cdot n}\cdot \mathbf{\Lambda}_{\hat{\psi}_\epsilon}(\hat{\varphi}_{\epsilon,\xi}(s)) &= \int_X (\sigma_\xi(s)^*\varphi(s)+\epsilon\sigma_\xi(s)^*\varphi_P-(\psi+\epsilon\varphi_P))(\sqrt{-1}\partial\bar\partial(\psi+\epsilon\varphi_P))^n\\ &= \int_X(\sigma_\xi(s)^*\varphi(s)-\psi)(\sqrt{-1}\partial\bar\partial\psi)^n\\ &\quad +\int_X(\sigma_\xi(s)^*\varphi(s)-\psi)[(\sqrt{-1}\partial\bar\partial(\psi+\epsilon\varphi_P))^n-(\sqrt{-1}\partial\bar\partial\psi)^n]\\ &\quad +\epsilon\int_X(\sigma_\xi(s)^*\varphi_P-\varphi_P)(\sqrt{-1}\partial\bar\partial(\psi+\epsilon\varphi_P))^n\\ &= (2\pi)^n L^{\cdot n}\cdot\mathbf{\Lambda}_\psi(\varphi_\xi(s))+\mathbf{I}_\epsilon(s)+\mathbf{II}_\epsilon(s).\end{aligned}$$

Note that

$$\Omega := \frac{1}{\epsilon}\left((\sqrt{-1}\partial\bar\partial(\psi+\epsilon\varphi_P))^n-(\sqrt{-1}\partial\bar\partial\psi)^n\right)\geq 0$$

and we have:

$$\mathbf{I}_\epsilon = \epsilon\int_X(\sigma_\xi(s)^*(\varphi(s)-\psi)+\sigma_\xi(s)^*\psi-\psi)\Omega=\epsilon(\mathbf{A}_\epsilon(s)+\mathbf{B}_\epsilon(s)).$$

Write $\mathbf{II}_\epsilon=\epsilon\mathbf{C}_\epsilon$. Then we get:

$$(2\pi)^n L_\epsilon^{\cdot n}\cdot\mathbf{\Lambda}'^{\infty}_{\hat{\psi}_\epsilon}(\Phi_{\epsilon,\xi}) = L^{\cdot n}\cdot\mathbf{\Lambda}'^{\infty}_\psi(\Phi_\xi)+\lim_{s\to+\infty}\frac{\epsilon\mathbf{A}_\epsilon(s)}{2s}+\epsilon\lim_{s\to+\infty}\frac{\mathbf{B}_\epsilon(s)}{2s}+\epsilon\lim_{s\to+\infty}\frac{\mathbf{C}_\epsilon(s)}{2s}.$$

Note that all of $\mathbf{\Lambda}_\psi(\varphi_\xi(s))$, $\mathbf{A}_\epsilon$, $\mathbf{B}_\epsilon$ and $\mathbf{C}_\epsilon$ are convex in $s$. Because $\epsilon\mathbf{A}_\epsilon$ is convex, $\epsilon\mathbf{A}_\epsilon\leq 0$ and $\lim_{\epsilon\to 0}\epsilon\mathbf{A}_\epsilon=0$, it is easy to verify that (see [46, Proof of Lemma 4.2])

$$\lim_{\epsilon\to 0}\lim_{s\to+\infty}\frac{\epsilon\mathbf{A}_\epsilon(s)}{2s}=0. \tag{207}$$

On the other hand, since $\{\hat{\psi}_\epsilon=\psi_0+\epsilon\psi_P\}$ are smooth, there exists $C>0$ independent of $\epsilon$ such that: $|\mathbf{B}'^{\infty}_\epsilon|\leq C, |\mathbf{C}'^{\infty}_\epsilon|\leq C$. Since $\lim_{\epsilon\to 0}L_\epsilon^{\cdot n}=L^{\cdot n}$, we indeed get:

$$\lim_{\epsilon\to+\infty}\mathbf{\Lambda}'^{\infty}_{\hat{\psi}_\epsilon}(\Phi_{\epsilon,\xi})=\mathbf{\Lambda}'^{\infty}_\psi(\Phi). \tag{208}$$

□

## 5.3 Step 3: Uniform convergence of $\mathbf{L}^{\rm NA}$ functions

We have the following identity:

$$K_Y=\mu^*(K_X+D)+\sum_{k=1}^g a_kE_k=\mu^*K_X-\sum_{i=1}^{g_1}b_iE_i''+\sum_{j=g_1+1}^{g}a_jE_j',$$

where for $i=1,\dots,g_1$, $E_i''=E_i$, $b_i=-a_i\in[0,1)$; and for $j=g_1+1,\dots,g$, $a_j>0$ and $E_j'=E_j$. Denote by $\lceil a_j\rceil$ the round up of $a_j$ and $\{a_j\}=\lceil a_j\rceil-a_j\in[0,1)$. Then we re-write the above identity as:

$$\begin{aligned}-K_Y+\sum_j\lceil a_j\rceil E_j' &= \mu^*(-K_X-D)+\sum_i b_iE_i''+\sum_j\{a_j\}E_j'\\ &= \frac{1}{1+\epsilon}\left((1+\epsilon)\mu^*(-K_X-D)-\epsilon\sum_i\theta_iE_i\right)\\ &\quad +\sum_i(b_i+\frac{\epsilon}{1+\epsilon}\theta_i)E_i''+\sum_j(\{a_j\}+\frac{\epsilon}{1+\epsilon}\theta_j)E_j'\\ &= \frac{1}{1+\epsilon}(\mu^*(-K_X-D)+\epsilon P)+\Delta_\epsilon,\end{aligned}$$

where $P = \mu^*(-K_X - D) - \sum_k \theta_k E_k$ and

$$\Delta_\epsilon = \sum_i b_i E_i'' + \sum_j \{a_j\} E_j' + \frac{\epsilon}{1+\epsilon} \sum_k \theta_k E_k = \Delta_0 + \frac{\epsilon}{1+\epsilon} E_\theta.$$

For simplicity of notations, we let $F := \sum_j \lceil a_j \rceil E_j'$. Then we have:

$$-K_Y + F = \frac{1}{1+\epsilon}(\mu^*(-K_X - D) + \epsilon P) + \Delta_\epsilon = \frac{1}{1+\epsilon}\hat{L}_\epsilon + \Delta_\epsilon = L_\epsilon + \Delta_\epsilon. \tag{209}$$

From now on, we denote:

$$B_\epsilon := \Delta_\epsilon - F = \sum_k \left(-a_k + \frac{\epsilon}{1+\epsilon}\theta_k\right) E_k. \tag{210}$$

Then we have the identity $-(K_Y + B_\epsilon) = L_\epsilon$. Note that the test configuration $(\mathcal{Y}_{\epsilon,m}, \mathcal{L}_{\epsilon,m})$ constructed in the above section induces a test configuration $(\mathcal{Y}_{\epsilon,m}, \mathcal{B}_{\epsilon,m}, \mathcal{L}_{\epsilon,m})$ of the pair $(Y, B_\epsilon)$.

Consider the Ding energy (28) associated to this decomposition. For any $\varphi_\epsilon \in (\mathcal{E}^1(L_\epsilon))^{\mathbb{K}}$, denote:

$$\mathbf{D}_{\psi_\epsilon}(\varphi_\epsilon) \quad = \quad -\mathbf{E}_{\psi_\epsilon}(\varphi_\epsilon) + \mathbf{L}_{(Y,B_\epsilon)}(\varphi_\epsilon)$$

where $\psi_\epsilon = \frac{\hat{\psi}_\epsilon}{1+\epsilon} = \frac{\psi_0 + \epsilon\varphi_P}{1+\epsilon}$ (see (198)) and (with $B = B_\epsilon = \Delta_\epsilon - F$ in (28)),

$$\mathbf{L}_{(Y,B_\epsilon)}(\varphi_\epsilon) = -\log\left(\int_Y e^{-\varphi_\epsilon} \frac{|s_F|^2}{|s_{\Delta_\epsilon}|^2}\right) =: \mathbf{L}_\epsilon(\varphi_\epsilon). \tag{211}$$

The following two results were proved in [46, 4.3]. The first one is based on [10, 6] and the second one based on [8, 13].

**Proposition 5.6.** *(1) With the above notations, let $\epsilon$ be sufficiently small such that $\lfloor \Delta_\epsilon \rfloor = 0$. Assume that $\Phi_\epsilon = \{\varphi_\epsilon(s)\}$ is a subgeodesic ray in $\mathcal{E}^1(Y, L_\epsilon)$. Then $\mathbf{L}_{(Y,B_\epsilon)}(\varphi_\epsilon(s))$ is convex in $s = \log|t|^{-1}$.*

*(2) Fix $0 \le \epsilon \ll 1$. Let $\Phi_\epsilon = \{\varphi_\epsilon(s)\}$ be a subgeodesic ray in $\mathcal{E}^1(Y, L_\epsilon)$ normalized such that $\sup(\varphi_\epsilon(s) - \psi_\epsilon) = 0$. We consider $\Phi_\epsilon$ as an $S^1$-invariant psh metric on $p_1'^* L_\epsilon \to Y_{\mathbb{C}}$. Then we have the identity:*

$$\lim_{s\to+\infty} \frac{\mathbf{L}_{(Y,B_\epsilon)}(\varphi_\epsilon(s))}{2s} = \inf_{\bar{w}\in\mathfrak{W}} (A_{Y_{\mathbb{C}}}(\bar{w}) - \bar{w}(\Phi_\epsilon) - \bar{w}((\Delta_\epsilon)_{\mathbb{C}}) + \bar{w}(F_{\mathbb{C}})) - 1, \tag{212}$$

*where $\mathfrak{W}$ is the set of $\mathbb{C}^*$-invariant divisorial valuations $w$ on $Y_{\mathbb{C}} = Y \times \mathbb{C}$ with $\bar{w}(t) = 1$.*

Now let $\hat{\Phi}_\epsilon$ be the same as in (198) and set $\Phi_\epsilon = \frac{1}{1+\epsilon}\hat{\Phi}_\epsilon$. To state the next result, we define functions on the set of valuations on $Y_{\mathbb{C}}$:

$$\begin{aligned} h_{\epsilon,m}(\bar{w}) \quad &:= \quad A_{Y_{\mathbb{C}}}(\bar{w}) - \frac{1}{1+\epsilon}\bar{w}(\hat{\Phi}_{\epsilon,m}) - \bar{w}((\Delta_\epsilon)_{\mathbb{C}}) + \bar{w}(F_{\mathbb{C}}) \\ &= \quad A_{Y_{\mathbb{C}}}(\bar{w}) - \frac{1}{1+\epsilon}\frac{1}{m}\bar{w}(\mathcal{J}(m\Phi)) - w((\Delta_\epsilon)_{\mathbb{C}}) + \bar{w}(F_{\mathbb{C}}) \end{aligned} \tag{213}$$

$$\begin{aligned} h_\epsilon(\bar{w}) \quad &:= \quad A_{Y_{\mathbb{C}}}(\bar{w}) - \frac{1}{1+\epsilon}\bar{w}(\Phi) - \bar{w}(\Delta_\epsilon)_{\mathbb{C}} + \bar{w}(F_{\mathbb{C}}) \\ &= \quad A_{Y_{\mathbb{C}}}(\bar{w}) - \bar{w}((\Delta_0)_{\mathbb{C}}) + \bar{w}(F_{\mathbb{C}}) - \frac{1}{1+\epsilon}\bar{w}(\Phi) - \frac{\epsilon}{1+\epsilon}\bar{w}((E_\theta)_{\mathbb{C}}). \end{aligned} \tag{214}$$

Then by (212) we have the identity:

$$\mathbf{I}_{\epsilon,m} := \mathbf{L}^{\mathrm{NA}}(\phi_{\epsilon,m}) = \mathbf{L}'^{\infty}(\Phi_{\epsilon,m}) = \inf_{\bar{w}\in\mathfrak{W}} h_{\epsilon,m}(\bar{w}) - 1, \quad \mathbf{I}_\epsilon := \mathbf{L}'^{\infty}(\Phi_\epsilon) = \inf_{\bar{w}\in\mathfrak{W}} h_\epsilon(\bar{w}) - 1. \tag{215}$$

**Proposition 5.7.** *There exists $K > 0$ such that if we set*

$$\mathfrak{W}_K := \{\bar{w} \in \mathfrak{W}; A_{Y_{\mathbb{C}}}(\bar{w}) < K\}, \tag{216}$$

*then the following statements are true:*

*(1) The following identities hold true:*

$$\mathbf{L}'^{\infty}(\Phi_\epsilon) = \inf_{\bar{w}\in\mathfrak{W}_K} h_\epsilon(\bar{w}), \quad \mathbf{L}^{\mathrm{NA}}(\phi_{\epsilon,m}) = \inf_{\bar{w}\in\mathfrak{W}_K} h_{\epsilon,m}(\bar{w}). \tag{217}$$

*(2) There exists a constant $C' > 0$ independent of $\epsilon$ and $m$ such that for any $\epsilon \geq 0$, $m \in \mathbb{N}$ and $\bar{w} \in \mathfrak{W}_K$, we have:*

$$|h_{\epsilon,m}(\bar{w}) - h_\epsilon(\bar{w})| \leq C'\frac{1}{m}, \quad |h_\epsilon(\bar{w}) - h_0(\bar{w})| \leq C'\epsilon. \tag{218}$$

*(3) The following limit hold true:*

$$\lim_{m\to+\infty} \mathbf{L}^{\mathrm{NA}}(\phi_{\epsilon,m}) = \lim_{s\to+\infty} \frac{\mathbf{L}_{(Y,B_\epsilon)}(\varphi_\epsilon(s))}{2s} =: \mathbf{L}'^{\infty}(\Phi_\epsilon). \tag{219}$$

$$\lim_{\epsilon\to 0} \mathbf{L}'^{\infty}(\Phi_\epsilon) = \mathbf{L}'^{\infty}(\Phi). \tag{220}$$

*Proof.* By the definition of multiplier ideals, for any $\bar{w} \in \mathring{\mathrm{Val}}(Y_{\mathbb{C}})$ we have:

$$\bar{w}(\mathcal{J}(m\Phi)) \leq m\,\bar{w}(\Phi) \leq \bar{w}(\mathcal{J}(m\Phi)) + A_{Y_{\mathbb{C}}}(\bar{w}). \tag{221}$$

So we get the following inequality for functions defined in (213) and (214):

$$h_\epsilon(\bar{w}) \leq h_{\epsilon,m}(\bar{w}) \leq h_\epsilon(\bar{w}) + \frac{1}{m}A_{Y_{\mathbb{C}}}(\bar{w}) \leq 2A_{\mathbb{C}}(\bar{w}) - \bar{w}((\Delta_0)_{\mathbb{C}}) + \bar{w}(F_{\mathbb{C}}).$$

So we there exists $C_1 > 0$ such that

$$\inf_{\bar{w}\in\mathfrak{W}} h_\epsilon(\bar{w}) \leq \inf_{\bar{w}\in\mathfrak{W}^{\mathbb{G}}} h_{\epsilon,m}(\bar{w}) \leq C_1. \tag{222}$$

Let $W_{\epsilon,m} := \{\bar{w} \in \mathfrak{W}; h_{\epsilon,m} \leq C_1 + 1\}$. Then

$$\mathbf{I}_\epsilon = \inf_{\bar{w}\in W_{\epsilon,m}} h_\epsilon(\bar{w}), \quad \mathbf{I}_{\epsilon,m} = \inf_{\bar{w}\in W_{\epsilon,m}} h_{\epsilon,m}(\bar{w}). \tag{223}$$

For any $\bar{w} \in W_{\epsilon,m}$, we have:

$$\begin{aligned} A_{Y_{\mathbb{C}}}(\bar{w}) &\leq C_1 + 1 + \bar{w}((\Delta_0)_{\mathbb{C}}) - \bar{w}(F_{\mathbb{C}}) + \frac{1}{1+\epsilon}\frac{1}{m}\bar{w}(\mathcal{J}(m\Phi)) + \frac{\epsilon}{1+\epsilon}\bar{w}((E_\theta)_{\mathbb{C}}) \\ &\leq C_1 + 1 + \bar{w}((\Delta_0)_{\mathbb{C}}) - \bar{w}(F_{\mathbb{C}}) + \frac{1}{1+\epsilon}\bar{w}(\Phi) + \frac{\epsilon}{1+\epsilon}\bar{w}((E_\theta)_{\mathbb{C}}) \\ &\leq C_1 + 1 + \bar{w}((\Delta_0)_{\mathbb{C}}) + \bar{w}(\Phi) + \bar{w}((E_\theta)_{\mathbb{C}}) \\ &\leq C_1 + 1 + C_2 + (1-\tau)A_{Y_{\mathbb{C}}}(\bar{w}). \end{aligned}$$

The last inequality is by [8, Lemma 5.5]. So if we let $K = \frac{C_1+1+C_2}{\tau}$, then $W_{\epsilon,m} \subseteq \mathfrak{W}_K$ for any $\epsilon, m$ and hence:

$$\mathbf{I}_\epsilon = \inf_{\bar{w}\in\mathfrak{W}_K} h_\epsilon(\bar{w}), \quad \mathbf{I}_{\epsilon,m} = \inf_{\bar{w}\in\mathfrak{W}_K} h_{\epsilon,m}(\bar{w}). \tag{224}$$

This proves the statement in (1).

Moreover, for any $\bar{w} \in \mathfrak{W}_K$ we then have:

$$h_\epsilon(\bar{w}) \leq h_{\epsilon,m}(\bar{w}) \leq h_\epsilon(\bar{w}) + \frac{K}{m}. \tag{225}$$

This proves the first estimate in (218). The second inequality was proved in [46, Proposition 4.6]. Finally the limits in (219)-(220) follows formally from (218).

□

The following proposition says that the infimum in (212) can be taken among $\mathbb{G}$-invariant valuations.

**Proposition 5.8.** *Let $\Phi_\epsilon = \{\varphi_\epsilon(s)\} \subset (\mathcal{E}^1(Y, L_\epsilon))^{\mathbb{K}} \times \mathbb{R}$ be as before. If we let $\mathfrak{W}^{\mathbb{G}}$ denote the set of $\mathbb{C}^* \times \mathbb{G}$ invariant divisorial valuations $\bar{w}$ on $Y \times \mathbb{C}$ with $\bar{w}(t) = 1$. Then we have:*

$$\mathbf{L}'^{\infty}(\Phi_\epsilon) = \inf_{\bar{w} \in \mathfrak{W}^{\mathbb{G}}} h_\epsilon(\bar{w}). \tag{226}$$

*Proof.* Note that $\Phi_{\epsilon,m}$ is associated to $\mathbb{C}^* \times \mathbb{G}$-equivariant test configuration $(\mathcal{Y}_{\epsilon,m}, \mathcal{B}_{\epsilon,m}, \mathcal{L}_{\epsilon,m})$. By choosing a $\mathbb{C}^* \times \mathbb{G}$-equivariant log resolutions in Remark 2.17 and arguing as in the proof of the above proposition, we see that the following infimum calculating $\mathbf{L}'^{\infty}(\Phi_{\epsilon,m})$ can be taken over $\mathfrak{W}^{\mathbb{G}} \cap \mathfrak{W}_K$:

$$\mathbf{L}'^{\infty}(\Phi_{\epsilon,m}) = \inf_{\bar{w} \in \mathfrak{W}} h_{\epsilon,m}(\bar{w}) = \inf_{\bar{w} \in \mathfrak{W}^{\mathbb{G}} \cap \mathfrak{W}_K} h_{\epsilon,m}(\bar{w})$$

For $\mathbf{L}'^{\infty}(\Phi_\epsilon)$, we can use (218) to estimate:

$$\left| \inf_{\bar{w} \in \mathfrak{W}^{\mathbb{G}} \cap \mathfrak{W}_K} h_\epsilon - \inf_{\bar{w} \in \mathfrak{W}^{\mathbb{G}} \cap \mathfrak{W}_K} h_{\epsilon,m} \right| \le C' \frac{1}{m} \tag{227}$$

So we can let $m \to +\infty$ and use (219) to conclude.

□

## 5.4 Step 4: Completion of the proof

With the above preparations, we can complete the proof of our main result. On the one hand, by (186),

$$\begin{aligned} \mathbf{L}'^{\infty}(\Phi) &= \lim_{s \to +\infty} \frac{\mathbf{L}(\varphi(s))}{2s} = \lim_{s \to +\infty} \frac{\mathbf{D}(\varphi(s))}{2s} + \lim_{s \to +\infty} \frac{\mathbf{E}(\varphi(s))}{2s} \\ &\le 0 - \mathbf{E}'^{\infty}(\Phi) = -1. \end{aligned} \tag{228}$$

Choose a sequence of divisorial valuations $v_k \in \mathring{\mathrm{Val}}(X)^{\mathbb{G}}$ such that

$$\mathbf{L}'^{\infty}(\Phi) \le A_{(X,D)}(v_k) - G(v_k)(\Phi)) < \mathbf{L}'^{\infty}(\Phi) + \frac{1}{k}, \tag{229}$$

and $A_{(X,D)}(v_k) \le K - 1$ where the constant $K$ is from Proposition 5.7. Note that $\mathbf{L}'^{\infty}(\Phi)$ is indeed finite by [8, Theorem 5.4].

By Corollary 3.23, there exist $\delta = \delta_{\mathbb{G}}(X, D) > 1$ and $\xi_k \in N_{\mathbb{R}}$ such that

$$A_{(X,D)}(v_{k,\xi_k}) \ge \delta S_L(v_{k,\xi_k}) \tag{230}$$

where $L = -K_X - D$. We claim that $|\xi_k|$ is uniformly bounded. To see this first recall that $\mathrm{Fut}_{(Z,D)} \equiv 0$ on $\mathfrak{t}$ under the assumption of $\mathbb{G}$-uniform Ding-stability. By using (130), we then have

$$\begin{aligned} 0 \le A_{(X,D)}(v_{k,\xi_k}) - \delta S_L(v_{k,\xi_k}) &= \delta(A_{X,D}(v_{k,\xi_k}) - S_L(v_{k,\xi_k})) - (\delta - 1) A_{(X,D)}(v_{k,\xi_k}) \\ &= \delta(A_{(X,D)}(v_k) - S_L(v_k)) - (\delta - 1) A_{(X,D)}(v_{k,\xi_k}). \end{aligned} \tag{231}$$

So we get the estimate:

$$A_{(X,D)}(v_{k,\xi_k}) \le \frac{\delta}{\delta - 1} A_{(X,D)}(v_k) \le \frac{\delta}{\delta - 1}(K - 1) = C_1.$$

This implies $|\xi_k| \le C_2$ for some $C_2$ independent of $k$. Indeed, we have $S_L(v_{k,\xi_k}) \le \delta^{-1}C_1$, which implies $\mathbf{\Lambda}^{\mathrm{NA}}(\mathcal{F}_{v_{k,\xi_k}}) \le (n+1)\delta^{-1}C_1$ (see (70)). By the proof of Lemma 3.15, we get $|\xi_k| \le C_2$ for some $C_2 > 0$ independent of $k$.

If $S_{L_0}(v_{k,\xi_k}) = 0$ then $v_{k,\xi_k}$ is trivial and $S_{L_\epsilon}(v_{k,\xi_k}) = 0$ for $\epsilon \ge 0$. Otherwise, $S_{L_\epsilon}(v_{k,\xi_k}) \ne 0$ for $0 \le \epsilon \ll 1$. Consider the quantity:

$$\Theta(\epsilon) := \frac{A_{(Y,B_\epsilon)}(E)(-K_Y - B_\epsilon)^{\cdot n}}{\int_0^\infty \mathrm{vol}_Y(-K_Y - B_\epsilon - x \cdot v_{k,\xi_k})dx}. \tag{232}$$

By the same calculation as in [46, 4.4], we get that there exists $C' > 0$ independent of $\epsilon$ and $v_{k,\xi}$ such that

$$\frac{\Theta(\epsilon)}{\Theta(0)} \ge 1 - C'\epsilon. \tag{233}$$

Set $\delta' := 1 + \frac{\delta-1}{2} > 1$. Then when $\epsilon$ is sufficiently small, we have

$$A_{(Y,B_\epsilon)}(v_{k,\xi_k}) = \Theta(\epsilon)\delta S_{L_\epsilon}(v_{k,\xi_k}) \ge (1-C'\epsilon)\delta S_{L_\epsilon}(v_{k,\xi_k}) \ge \delta' S_{L_\epsilon}(v_{k,\xi_k}). \tag{234}$$

Now we can estimate as follows:

$$\begin{aligned}
& \mathbf{L}^{\mathrm{NA}}_{(Y,B_\epsilon)}(\phi_{\epsilon,m}) + O(\epsilon, m^{-1}, k^{-1}) \\
= \; & A_{(Y,B_\epsilon)}(v_k) + \phi_{\epsilon,m}(v_k) \qquad \text{(by (229) and Proposition 5.7 )} \\
= \; & A_{(Y,B_\epsilon)}(v_{k,\xi_k}) + \phi_{\epsilon,m,-\xi_k}(v_{k,\xi_k}) \qquad \text{(by (123))} \\
\ge \; & \delta' S_{L_\epsilon}(v_{k,\xi_k}) + \phi_{\epsilon,m,-\xi_k}(v_{k,\xi_k}) \qquad \text{(by (234))} \\
= \; & \delta'(S_{L_\epsilon}(v_{k,\xi_k}) + \delta'^{-1}\phi_{\epsilon,m,-\xi_k}(v_{k,\xi_k})) \qquad \text{(note } \delta' > 1) \\
\ge \; & \delta' \mathbf{E}^{\mathrm{NA}}_{L_\epsilon}(\delta'^{-1}\phi_{\epsilon,m,-\xi_k}) \qquad \text{(by (176) or [16, Proposition 7.5])} \\
= \; & \left(\delta' \mathbf{J}^{\mathrm{NA}}_{L_\epsilon}(\delta'^{-1}\phi_{\epsilon,m,-\xi_k}) - \mathbf{J}^{\mathrm{NA}}_{L_\epsilon}(\phi_{\epsilon,m,-\xi_k})\right) + \mathbf{E}^{\mathrm{NA}}_{L_\epsilon}(\phi_{\epsilon,m,-\xi_k}) \qquad \text{(by (90))} \\
\ge \; & (1-\delta'^{-1/n})\mathbf{J}^{\mathrm{NA}}_{L_\epsilon}(\phi_{\epsilon,m,-\xi_k}) + \mathbf{E}^{\mathrm{NA}}_{L_\epsilon}(\phi_{\epsilon,m,-\xi_k}) \qquad \text{(by [16, Lemma 6.17])} \\
= \; & (1-\delta'^{-1/n})(\mathbf{\Lambda}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,m,-\xi_k}) - \mathbf{E}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,m,-\xi_k})) + \mathbf{E}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,m,-\xi_k}) \quad \text{(by Proposition 2.18)} \\
= \; & (1-\delta'^{-1/n})\mathbf{\Lambda}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,m,-\xi_k}) + \delta'^{-1/n}\mathbf{E}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,m,-\xi_k}) \\
\ge \; & (1-\delta'^{-1/n})\mathbf{\Lambda}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,-\xi_k}) + \delta'^{-1/n}\mathbf{E}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,-\xi_k}) \qquad \text{(by (202) – (203))} \\
= \; & (1-\delta'^{-1/n})\mathbf{J}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,-\xi_k}) + \mathbf{E}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,-\xi_k}).
\end{aligned}$$

Letting $m \to +\infty$ and using (219), we get the following inequality:

$$\mathbf{L}'^{\infty}_{(Y,B_\epsilon)}(\Phi_\epsilon) + O(k^{-1}) \ge (1-\delta'^{-1/n})\mathbf{J}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,-\xi_k}) + \mathbf{E}'^{\infty}_{\psi_\epsilon}(\Phi_{\epsilon,-\xi_k}).$$

Letting $\epsilon \to 0$ and using (220), (204)-(205), we get:

$$\begin{aligned}
\mathbf{L}'^{\infty}(\Phi) + O(k^{-1}) \;\; & \ge \;\; (1-\delta'^{-1/n})\mathbf{J}'^{\infty}(\Phi_{-\xi_k}) + \mathbf{E}'^{\infty}(\Phi_{-\xi_k}) \\
& = \;\; (1-\delta'^{-1/n})\mathbf{J}'^{\infty}(\Phi_{-\xi_k}) + \mathbf{E}'^{\infty}(\Phi) \\
& \ge \;\; (1-\delta'^{-1/n})\chi - 1. \quad \text{( by Corollary 5.3 )}
\end{aligned}$$

But when $k \gg 1$, this contradicts (228) because $\chi > 0$.

**Remark 5.9.** *Corresponding to Remark 5.4, the above contradiction chain can be simplified if we know the expected inequality* $\inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}'^{\infty}(\Phi_\xi) > 0$ *is true.*

**Remark 5.10.** *In the above proof, if $X$ is already smooth, then we can set $(Y, B) = (X, \emptyset)$ to give a proof of Hisamoto's claimed result. However, even in this case, our argument above is quite different with Hisamoto's argument. More specifically, we have the following comments about his proof which does not seem to be complete:*

(1) *Hisamoto's argument does not use Mabuchi-energy. However currently it seems not enough to use just Ding energy to bound the entropy in order to to apply compactness result Theorem 2.11 (from [6]). In fact, the Legendre transform only gives "* $\leq$ *" for the second identity in the formula after [40, Theorem 4.1].*

(2) *[40, Lemma 4.3] claims that*

$$\mathbf{\Lambda}'^{\infty}(\sigma_\xi(s)^*\psi) = \max\left\{\kappa_i^{(m)} := \frac{\langle \alpha_i^{(m)}, \xi\rangle}{m}; i = 1, \dots, N_m\right\} = 0$$

*implies the identity:*

$$\mathbf{\Lambda}^{\rm NA}(\mathcal{X}_{m,\xi}, \mathcal{L}_{m,\xi}) = \max\left\{\frac{\lambda_i^{(m)} + \kappa_i^{(m)}}{m}; i = 1, \dots, N_m\right\} = 0.$$

*This is in general not true. In our argument, we don't use this and, instead use crucially the monotonicity of* $\mathbf{\Lambda}$*-energy.*

(3) *The contradiction at the end of the paper [40] needs the inequality (with his notation and* $\eta = 0$ *in our case):*

$$\langle \mu_m, 1 + \eta\rangle + E^{\rm NA}(\mathcal{X}_m, \mathcal{L}_m) < 0.$$

*But this seems not clear. In our argument, a key point is that* $\mathbf{J}'^{\infty}(\Phi_{-\xi_k}) \geq \chi > 0$ *for all* $v_k$*.*

# A Some properties of reductive groups


## Jun Yu[1]


**Proposition A.1.** *Let $G$ be a connected reductive complex Lie group and $K$ be a maximal compact subgroup of $G$. Then we have $N_G(K) = C(G) \cdot K = C(G)_0 \cdot K$ where $C(G)_0$ is the identity component of the center $C(G)$ of $G$.*

*Proof.* Write $G = C(G)_0 \cdot G_1 \cdot G_2 \cdots G_s$ where $G_1, \dots, G_s$ are simple factors of $G$. Write $K_i = K \cap G_i$, $K_0 = K \cap C(G)_0$. Then $K = K_0 \cdot K_1 \cdots K_s$ and each $K_i$ is a maximal compact subgroup of $G_i$ $(1 \leq i \leq s)$. Clearly $C(G)_0 \subset N_G(K)$.

Conversely, if $g = g_1 \cdot g_2 \cdots g_s$ normalizes $K$, then each $g_i$ normalizes $K_i$. Hence it suffices to show that $N_{G_i}(K_i) = K_i$ for each $i (1 \leq i \leq s)$. By this discussion, we may assume that $G$ itself is simple. Write $H = N_G(K)$. Then $H$ is a closed subgroup of $G$, and $K$ is a normal subgroup of $H$.

Since $G$ is assumed to be simple, the only Lie subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$ contains $\mathfrak{k} = \mathrm{Lie}(K)$ are $\mathfrak{g}$ and $\mathfrak{k}$. Thus $\mathfrak{h} = \mathrm{Lie}(H) = \mathfrak{g}$ or $\mathfrak{k}$. When $\mathfrak{h} = \mathfrak{g}$, then $H = G$ which is impossible.

When $\mathfrak{h} = \mathfrak{k}$, $H$ is also compact. Then for any $x \in H$, $\mathrm{Ad}(x) \in \mathrm{GL}(\mathfrak{g})$ is elliptic (i.e. eigenvalues of $\mathrm{Ad}(x)$ all have norm 1). On the other hand, we have the Cartan decomposition $G = K \exp(\mathfrak{p}_0)$ where $\mathfrak{p}_0$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Since for any $g \in \exp(\mathfrak{p}_0)$, $\mathrm{Ad}(g)$ has positive real eigenvalues, $H \cap \exp(\mathfrak{p}_0) = 1$. Then

$$H = H \cap G = H \cap K \exp(\mathfrak{p}_0) = K \cap (H \cap \exp(\mathfrak{p}_0)) = K.$$

□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**Proposition A.2.** *Let $G$ be a connected complex reductive Lie group, and $K_1, K_2$ be two maximal compact subgroups. Assume that $K_1, K_2$ have a common maximal torus $T$. Set $T_{\mathbb{C}} = C_G(T)$ which is a maximal torus of $G$. Then the following hold true:*

*(1) $K_2 = tK_1t^{-1} =: \mathrm{Ad}(t)K_1$ for some $t \in T_{\mathbb{C}}$.*

*(2) If $K_2 = \mathrm{Ad}(t)K_1$, then $K_1 = K_2$ if and only if $t \in T$.*

*Proof.* (1) It is well-known that any two maximal compact subgroups of $G$ are conjugate. Thus there exists $g \in G$ such that $K_2 = \mathrm{Ad}(g)K_1$. Then $\mathrm{Ad}(g)T$ and $T$ are maximal tori of $K_2$. Hence there exists $k_2 \in K_2$ such that $\mathrm{Ad}(g)T = \mathrm{Ad}(k_2)T$. Set $g' = k_2^{-1}g$. Then

$$\mathrm{Ad}(g')K_1 = \mathrm{Ad}(k_2)\mathrm{Ad}(g)K_1 = \mathrm{Ad}(k_2^{-1})K_2 = K_2$$

and

$$\mathrm{Ad}(g')T = \mathrm{Ad}(k_2^{-1})\mathrm{Ad}(g)T = \mathrm{Ad}(k_2)^{-1}\mathrm{Ad}(k_2)T = T.$$

Thus $g' \in N_G(T)$. It is well-known that $T_{\mathbb{C}} := C_G(T)$ is a maximal torus of $G$ and

$$N_G(T) = N_{K_2}(T) \cdot T_{\mathbb{C}}.$$

Write $g' = n \cdot t$ for $n \in N_{K_2}(T)$ and $t \in T_{\mathbb{C}}$. Then

$$K_2 = \mathrm{Ad}(n^{-1})K_2 = \mathrm{Ad}(n^{-1})\mathrm{Ad}(g')K_1 = \mathrm{Ad}(n^{-1}g')K_1 = \mathrm{Ad}(t)K_1.$$

(2) Set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{t}_{\mathbb{C}} = \mathrm{Lie}(T_{\mathbb{C}})$. Then one has a root space decomposition:

$$\mathfrak{g} = \mathfrak{t}_{\mathbb{C}} \bigoplus \left( \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right),$$

where $\Delta = \Delta(\mathfrak{g}, \mathfrak{t}_{\mathbb{C}})$ are roots of $\mathfrak{g}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{g}_\alpha$ is the root space of $\alpha$. It is well-known that each $\mathfrak{g}_\alpha$ has dimension one. Chose $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ for any $\alpha \in \Delta$. Choose a positive system $\Delta^+ \subset \Delta$. It is well-known that

$$\mathfrak{k}_1 := \mathrm{Lie}(K_1) = \mathfrak{t} \bigoplus \left( \bigoplus_{\alpha \in \Delta^+} \left( \mathbb{R}(X_\alpha + a_\alpha X_{-\alpha}) \oplus \mathbb{R}\mathbf{i}(X_\alpha + b_\alpha X_\alpha) \right) \right) \tag{235}$$

for some constants $a_\alpha, b_\alpha \in \mathbb{C}^\times$ with $a_\alpha \neq b_\alpha$.

Set $\mathfrak{a}$ to be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{t}_{\mathbb{C}}$ and $A = \exp(\mathfrak{a})$. Then $T_{\mathbb{C}} = AT$. Assume $\mathrm{Ad}(t)K_1 = K_1$. Clearly $\mathrm{Ad}(t_1)K_1 = K_1$ for $t_1 \in T \subset K_1$. So one may assume that $t = a \in A$. For any $\alpha \in \Delta^+$, $\alpha(a) > 0$. Then the Lie algebra of $\mathrm{Ad}(t)K_1 = \mathrm{Ad}(a)K_1$ is equal to:

$$\mathfrak{t} \bigoplus \left( \bigoplus_{\alpha \in \Delta^+} \left( \mathbb{R}(X_\alpha + a_\alpha \alpha(a)^{-2} X_{-\alpha}) \oplus \mathbb{R}\mathbf{i}(X_\alpha + b_\alpha \alpha(a)^{-2} X_{-\alpha}) \right) \right). \tag{236}$$

For it to be equal to $\mathfrak{k}_1$, one must have $\alpha(a)^{-2} = 1$ for all $\alpha \in \Delta^+$. Then $a = 1$.

□

**Proposition A.3.** *Let $G$ be a connected complex reductive Lie group, and $K_1, K_2$ be two maximal compact subgroups. Assume that $K_1, K_2$ have a common compact subgroup $K$ that in turn contains a maximal compact torus $T$ of $G$. Then $K_2 = tK_1t^{-1}$ for some $t \in C(K_{\mathbb{C}})$ (the center of $K_{\mathbb{C}}$).*

*Proof.* We use the same notations as in the proof of the last proposition. By Proposition A.2, there exists $t \in T_{\mathbb{C}}$ such that $K_2 = tK_1t^{-1}$. We just need to show that $t \in C(K_{\mathbb{C}})$. Similar to (235), we have the decomposition

$$\mathfrak{k} := \mathrm{Lie}(K) = \mathfrak{t} \bigoplus \left( \bigoplus_{\alpha \in \Delta'^{+}} \left( \mathbb{R}(X_\alpha + a_\alpha X_{-\alpha}) \oplus \mathbb{R}\mathbf{i}(X_\alpha + b_\alpha X_\alpha) \right) \right),$$

where $\Delta'^{+}$ is a positive system for $\mathrm{Lie}(K_{\mathbb{C}})$ with respect to $\mathfrak{t}_{\mathbb{C}}$. Because $K_1 \subseteq K$, $\mathfrak{k}$ embeds into $\mathfrak{k}_1$ via the inclusion $\Delta'^{+} \subseteq \Delta^{+}$. By using the expression in (236), we see that the Lie algebra of $K_2 = \mathrm{Ad}(t)K_1$ contains $\mathrm{Lie}(K)$ if and only if $\alpha(a)^{-2} = 1$ for all $\alpha \in \Delta'^{+}$. This holds if and only if $t \in C(K_{\mathbb{C}})$.

□

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