

EXPLICIT STILLMAN BOUNDS FOR ALL DEGREES

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ABSTRACT.

1. INTRODUCTION

Let K be a field, $R = K[x_1, \dots, x_N]$ be a polynomial ring over K , and I be an ideal of R generated by n forms of degrees d_1, \dots, d_n . We will denote the projective dimension of a module over R by $\text{pd}_R(M)$. Stillman (see [21]) conjectured that the projective dimension of I can be bounded in terms of n and d_1, \dots, d_n but independent of N . We will refer to such bounds as Stillman bounds. Ananyan and Hochster were the first to give an affirmative answer to Stillman's conjecture in [1], where they showed the existence of Stillman bounds by proving the existence of small subalgebras and small subalgebra bounds nB (defined in [1, Theorem B] or Theorem 2.2). Stillman's conjecture was later reproved in [12] and [17], both using topological Noetherianity results from [11].

With the existence proven, the next question is to find explicit Stillman bounds. While many early and recent works [2] [3] [4] [5] [8] [16] [15] [18] [19] [20] have established Stillman bounds for degree 4 or less, the question for degree 5 and higher remains untouched. We attempt to give explicit Stillman bounds for all degrees. We notice that if we prove a bound called $D(k, d)$, which controls the number of generators of a minimal prime over an ideal of a regular sequence of k or fewer polynomials with degree d , and supplements $D(k, d)$ into Ananyan and Hochster's proof in [1], then theoretically we can make their proof into a recursive algorithm to obtain explicit Stillman bounds for all degrees.

We prove the following lemma which establish an effective value for $D(k, d)$. In [1], Ananyan and Hochster only showed the existence of $D(k, d)$. Notice that the function ${}^nB(k, d)$ is defined in Corollary 2.3 of Section 2.

Lemma 3. *Let K be an algebraically closed field, $P \subset K[x_1, \dots, x_N]$ be a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree at most d . Then the minimal number of generators of P is bounded by*

$$D(k, d) = (2d)^{2^{{}^1B(k, d)-1}}.$$

Notice that in the above lemma, the assumption of the field being algebraically closed is needed only for existence of ${}^1B(k, d)$ as in [1, Theorem B], and the proof still works if we drop this assumption and replace ${}^1B(k, d)$ by $\text{pd}_R(R/P)$ or N (see Theorem 3.5).

However, after a closer examination of [2], we realize that the computational complexity of Ananyan and Hochster's inductive proof would make the bound too large to be meaningful as d gets larger. We explain the reason in Subsection 3. Moreover we give an estimate of the bound

Date: December 4, 2021.

2010 Mathematics Subject Classification. Primary 13D02, 13F20, 13P20.

Key words and phrases. Stillman bound, projective dimension, regular sequence, R_η .

${}^1\mathcal{B}(n, d)$ for degree 2, 3 and a general description for degree 4, to give the reader a feeling about its bad behavior.

This section is organized as follows. In Section 2, we first state the small subalgebra theorems of [1] after giving their necessary definitions, then set up notations needed for the next section. In Section 3, we construct the bound $D(k, d)$, discuss how we insert the bound $D(k, d)$ into Ananyan and Hochster's proof, and conclude the section with a theorem establishing our bounds for nB with a recurrence relation.

2. NOTATION

We first recall some definitions in [1, §1] which are needed for stating the theorems in [1]. Let $R = K[x_1, \dots, x_N]$ be a polynomial ring over a field K . Let V be a finite dimensional graded vector space of R , then we say V has *dimension sequence* $\delta = (\delta_1, \dots, \delta_d)$ if $V = V_1 \oplus \dots \oplus V_d$ as a direct sum of its graded components with $\dim_K V_i = \delta_i$.

A function of several variables is called *ascending* if it is increasing in any one variable while the other variables are fixed.

A form $F \in R$ has a *k-collapse* if it can be written as a graded combination of k or fewer forms of strictly smaller positive degree. We say F has *strength k* if it has a $k+1$ -collapse but no k -collapse. By convention we set the strength of a linear form to be $+\infty$.

A sequence of elements G_1, \dots, G_s in a Noetherian ring R is a *prime sequence* (respectively, an R_η -sequence) if for $0 \leq i \leq s$, $R/(G_1, \dots, G_i)$ is a domain (respectively, satisfies the Serre condition R_η). When R is a polynomial ring, for any $\eta \geq 1$ an R_η -sequence is a prime sequence and hence a regular sequence.

Our goal is to give explicit bounds to the functions nA and nB for any degree, which are defined in Theorem A, Theorem B, and Corollary B of [1].

Theorem 2.1 (Ananyan-Hochster [1]). *There are ascending functions $A = (A_1, \dots, A_d)$ and, for every integer $\eta \geq 1$, ${}^nA = ({}^nA_1, \dots, {}^nA_d)$ from dimension sequences $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{N}^d$ to \mathbb{N}^d with the following property: For every algebraically closed field K and every positive integer N , if $R = K[x_1, \dots, x_N]$ is a polynomial ring, and V denotes a graded K -vector subspace of R of vector space dimension n with dimension sequence $(\delta_1, \dots, \delta_d)$, such that for $1 \leq i \leq d$, the strength of every nonzero element of V_i is at least $A_i(\delta)$ (respectively, ${}^nA_i(\delta)$), then every sequence of K -linearly independent forms in V is a regular sequence (respectively, is an R_η -sequence).*

Theorem 2.2 (Ananyan-Hochster [1]). *There is an ascending function B from dimension sequences $\delta = (\delta_1, \dots, \delta_d)$ to \mathbb{Z}_+ with the following property. If K is an algebraically closed field and V is a finite-dimensional \mathbb{Z}_+ -graded K -vector subspace of a polynomial ring R over K with dimension sequence δ , then V (and, hence, the K -subalgebra of R generated by V) is contained in a K -subalgebra of R generated by a regular sequence G_1, \dots, G_s of forms of degree at most d , where $s \leq B(\delta)$. Moreover, for every $\eta \geq 1$ there is such a function nB with the additional property that every sequence consisting of linearly independent homogeneous linear combinations of the elements in G_1, \dots, G_s is an R_η -sequence.*

The next corollary, as remarked in [1], follows immediately by taking ${}^n\mathcal{B}(n, d)$ to be the maximum of ${}^nB(\delta)$ over all dimension sequences with at most d entries and sum of entries at most n .

Corollary 2.3 (Ananyan-Hochster [1]). *There is an ascending function ${}^n\mathcal{B}(n, d)$, independent of K and N , such that for all polynomial rings $R = K[x_1, \dots, x_N]$ over an algebraically closed field*

K and all graded vector subspaces V of R of dimension at most n whose homogeneous elements have positive degree at most d , the elements of V are contained in a subring $K[G_1, \dots, G_B]$, where $B \leq {}^n\mathcal{B}(n, d)$ and G_1, \dots, G_B is an R_n -sequence of forms of degree at most d .

We next introduce notations that are needed for the proof of Lemma 3.2 and Lemma 3.3. For a finitely generated graded R -module M , let $\beta_{ij}(M)$ be the graded Betti numbers of M and $\beta_i(M) = \sum_j \beta_{ij}(M)$ be the i -th Betti number of M . The Castelnuovo-Mumford regularity of M is defined as $\text{reg}(M) = \max_{i,j} \{j - i : \beta_{ij}(M) \neq 0\}$.

Let J be a monomial ideal. We say J is *strongly stable* if for each monomial u of J , $x_i|u$ implies $x_j u / x_i \in J$ for each $j < i$. Let $G(J)$ be the set of minimal monomial generators of J and $D(J)$ be the largest degree of monomials in $G(J)$. If u is a monomial, let $m(u) := \max\{i : x_i|u\}$. By the Eliahou-Kervaire resolution in [14], if J is strongly stable then $\beta_i(J) = \sum_{u \in G(J)} \binom{m(u)-1}{i}$.

Let I be a monomial ideal in $K[x_1, \dots, x_N]$ where K is an infinite field, we may assume I is generated by monic monomials, if K' is any other field then let $I_{K'}$ be the ideal generated by the image of these monomials in $K'[x_1, \dots, x_N]$. Let $\text{gin}_{\text{rlex}}(I)$ be the generic initial ideal of I with respect to the degree reverse lexicographical order. The *zero-generic initial ideal* of I with respect to the degree reverse lexicographical order is defined to be

$$\text{Gin}_0(I) := (\text{gin}_{\text{rlex}}((\text{gin}_{\text{rlex}}(I))_{\mathbb{Q}}))_K.$$

The zero-generic initial ideal is explored in more details in [7]. We need this notion in §3 to treat the positive characteristic cases. Notice that in characteristic 0, the zero-generic initial ideal is equal to the usual generic initial ideal.

3. PROCEDURES TO REALIZE THE BOUND VIA A RECURSIVE ALGORITHM

We start this section by constructing a bound, denoted $D(k, d)$ in [1], for the number of generators of a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree d . We note that this bound is independent of the number of variables in the polynomial ring, which is necessary for our purpose. The following lemma, see [9, Theorem 27], tells us that the above minimal prime can be written as the ideal of the regular sequence colon by a form with bounded degree, which is key to the proof of Lemma 3.2.

Lemma 3.1 (Chardin [9]). *Let $P \subset K[x_1, \dots, x_N]$ be a minimal prime of an ideal generated by a homogeneous regular sequence f_1, \dots, f_k of degrees d_1, \dots, d_k . There exists a form f of degree at most $d_1 + \dots + d_k - k$ such that*

$$P = (f_1, \dots, f_k) : (f).$$

The next two lemmas justify the fact that we can choose $D(k, d)$ to be $(2d)^{2^{{}^1\mathcal{B}(k, d)} - 1}$. Assuming the number of variables is known, we show in Lemma 3.2 how to bound the minimal number of generators of a particular kind of colon ideals as in Lemma 3.1. In pursuance of an optimal bound, we extensively apply results and proofs of [6] and [7]. For an alternate way to obtain a value for $D(k, d)$, we refer the reader to Remark 3.4, which could still be used to construct explicit but larger bounds for ${}^n\mathcal{B}(\delta)$.

Lemma 3.2. *Let $I \subset K[x_1, \dots, x_{B+1}]$ be an ideal generated by a regular sequence of c forms of degree at most d , and f be a form of degree at most $cd - c$. Then the minimal number of*

generators of $I : (f)$ is bounded by

$$\beta_0(I : (f)) \leq B(d+1)(2d) \prod_{i=3}^B ((d^2 + 2d - 1)^{2^{i-3}} + 1) + 1.$$

If we further assume $B, d \geq 3$ or $B \geq 4$, then we have the additional inequality

$$\beta_0(I : (f)) \leq (2d)^{2^{B-1}}.$$

Proof. If $c = 1$ then $\beta_0(I : (f)) = 1$, so assume $c \geq 2$. After a faithfully flat base change, we may assume K is infinite. Denote $R := K[x_1, \dots, x_{B+1}]$ and let f_1, \dots, f_c be the regular sequence with $\deg(f_i) \leq d$. Consider the exact sequence

$$0 \rightarrow R/(I : (f)) \xrightarrow{f} R/I \rightarrow R/(I + (f)) \rightarrow 0.$$

Since $c \leq \beta_0(I + (f)) \leq c + 1$, using the long exact sequence of $\mathrm{Tor}_i^R(-, K)$ induced from the above short exact sequence, we get $\beta_0(I : (f)) \leq \beta_1(I + (f)) + 1$.

With the notations of Section 2, let $J := \mathrm{Gin}_0(I + (f))$. Denote $R_{[i]} := K[x_1, \dots, x_i]$. Let $(I + (f))_{\langle i \rangle}$ denote the image of $I + (f)$ in $R/(l_{B+1}, \dots, l_{i+1}) \cong R_{[i]}$ where l_{B+1}, \dots, l_{i+1} are general linear forms, and let $m_{[i]}$ denotes the homogeneous maximal ideal of $R_{[i]}$. Let $J_{[i]}$ denote $J \cap R_{[i]}$. By [2, Proposition 2.2], J is strongly stable with $\beta_1(I + (f)) \leq \beta_1(J)$. So by [14] and [6, Proposition 1.6], $\beta_1(J) = \sum_{u \in G(J)} (m(u) - 1) \leq B \cdot |G(J)| \leq B \prod_{i=1}^B (D(J_{[i]}) + 1)$. Using [7, Theorem 2.20], we can get $D(J_{[i]}) \leq \mathrm{reg}((I + (f))_{\langle i \rangle})$ for all i . Notice that $\mathrm{reg}((I + (f))_{\langle i \rangle}) \leq id - i + 1$ for all $i \leq c$, because $m_{[i]}^{id-i+1} \subseteq (I + (f))_{\langle i \rangle}$.

To bound $\mathrm{reg}((I + (f))_{\langle i \rangle})$ for $i \geq c + 1$, we follow the proof of [6, Theorem 2.4 and Corollary 2.6]. Let $\lambda(M)$ denote the length of an Artinian module M . Using the same proof of [6, Theorem 2.4], we can get

$$\begin{aligned} \mathrm{reg}((I + (f))_{\langle i \rangle}) &\leq \max\{d, cd - c, \mathrm{reg}((I + (f))_{\langle i-1 \rangle})\} \\ &\quad + \lambda\left(\frac{R_{[c]}}{(I + (f))_{\langle c \rangle}}\right) \prod_{j=c+2}^i \mathrm{reg}((I + (f))_{\langle j-1 \rangle}) \\ &\leq \max\{d, cd - c, \mathrm{reg}((I + (f))_{\langle i-1 \rangle})\} + d^c \prod_{j=c+1}^{i-1} \mathrm{reg}((I + (f))_{\langle j \rangle}). \end{aligned} \tag{3.1}$$

The last inequality holds since $R_{[c]}/(I + (f))_{\langle c \rangle}$ is a quotient ring of $R_{[c]}/(g_1, \dots, g_c)$, where g_1, \dots, g_c is the image of f_1, \dots, f_c in $R_{[c]}$ and is a regular sequence with $\deg(g_i) \leq d$.

Now we use (3.1) recursively to bound $\mathrm{reg}((I + (f))_{\langle i \rangle})$ for $i \geq c + 1$. Set $B_0 := cd - c + 1$, recall that B_0 bounds $\mathrm{reg}((I + (f))_{\langle c \rangle})$. Apply 3.1 to $(I + (f))_{\langle c+1 \rangle}$ to get $\mathrm{reg}((I + (f))_{\langle c+1 \rangle}) \leq cd - c + 1 + d^c =: B_1$. For $j \geq 2$, we set $B_j := B_{j-1} + d^c \prod_{k=1}^{j-1} B_k \leq (B_{j-1})^2 \leq (B_1)^{2^{j-1}}$. Hence for all $i \geq c + 1$, $\mathrm{reg}((I + (f))_{\langle i \rangle}) \leq B_{i-c} \leq (cd - c + 1 + d^c)^{2^{i-c-1}} \leq (d^2 + 2d - 1)^{2^{i-3}}$, where the last inequality holds since the second last bound is decreasing as a function of c and $c \geq 2$.

If $B \geq 4$, then $B \leq 2^{B-2}$. Combining all the previous inequalities, we get

$$\begin{aligned} \beta_0(I : (f)) &\leq B(d+1)(2d) \prod_{i=3}^B ((d^2 + 2d - 1)^{2^{i-3}} + 1) + 1 \\ &\leq 2^{B-2}(d+1)(2d) \prod_{i=3}^B (d^2 + 2d)^{2^{i-3}} \leq (2d)(2d) \prod_{i=3}^B (2d^2 + 4d)^{2^{i-3}} \\ &\leq (2d)^2 \prod_{i=3}^B (2d)^{2^{i-2}} = (2d)^{2^{B-1}}. \end{aligned}$$

If $B = 3$ and $d \geq 3$, one easily checks that $\beta_0(I : (f)) \leq 3(d+1)(2d)(d^2 + 2d) + 1 \leq (2d)^4$. \square

Lemma 3.3 constructs our value $(2d)^{2^{1\mathcal{B}(k,d)-1}}$ for the bound $D(k, d)$ by passing to a polynomial subring with at most $1\mathcal{B}(k, d) + 1$ many variables first, using Corollary 2.3, then combining the result of Lemma 3.1 and Lemma 3.2.

Lemma 3.3. *Let K be an algebraically closed field, $P \subset K[x_1, \dots, x_N]$ be a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree at most d . Then the minimal number of generators of P is bounded by*

$$\beta_0(P) \leq (2d)^{2^{1\mathcal{B}(k,d)-1}}.$$

Proof. Let f_1, \dots, f_c be the regular sequence with $c \leq k$ and $\deg(f_i) \leq d$, let I be the ideal it generates. By Corollary 2.3, there exists a prime sequence G_1, \dots, G_s with $s \leq 1\mathcal{B}(k, d)$ such that $f_1, \dots, f_c \in K[G_1, \dots, G_s]$. Denote $R = K[x_1, \dots, x_N]$ and $S = K[G_1, \dots, G_s]$. Then $\text{pd}_R(R/I) = c \leq s$ and $\text{pd}_S(S/P \cap S) \leq s$. Notice that R is a free and thus faithfully flat module over S since we can extend G_1, \dots, G_s to a maximal regular sequence G_1, \dots, G_N in R to get free extensions $K[G_1, \dots, G_s] \hookrightarrow K[G_1, \dots, G_N]$ and $K[G_1, \dots, G_N] \hookrightarrow K[x_1, \dots, x_N]$. Consequently we get $\text{pd}_R(R/P) \leq s$ once we have shown $P = (P \cap S)R$. By faithfully flatness $f_1, \dots, f_c \in P \cap S$ remains a regular sequence in S and so $c = \text{ht } P \geq \text{ht } (P \cap S)R = \text{ht } P \cap S \geq c$. Now by [1, Corollary 2.9], $(P \cap S)R$ is a prime ideal, therefore $P = (P \cap S)R$.

By Lemma 3.1, there exists a form $f \in R$ of degree at most $cd - c$ such that $P = I : (f)$. Consider the exact sequence

$$0 \rightarrow R/P \xrightarrow{f} R/I \rightarrow R/(I + (f)) \rightarrow 0. \quad (3.2)$$

It follows that $\text{pd}_R(R/(I + (f))) \leq s + 1$. Then $\text{depth } R/(I + (f)) \geq N - (s + 1)$ by the Auslander-Buchsbaum formula. Let $l_{s+2}, \dots, l_N \in R$ be a sequence of linear forms regular on R/P , $R/(I + (f))$, and $R/(f_1, \dots, f_c)$. Fix a graded isomorphism from $R/(l_{s+2}, \dots, l_N)$ to $K[x_1, \dots, x_{s+1}]$, let \bar{R} denote $K[x_1, \dots, x_{s+1}]$ and “ $-$ ” denote the image of polynomials or ideals of R in \bar{R} . Notice that $\beta_0(P) = \beta_0(\bar{P})$. Since l_{s+2}, \dots, l_N is a regular sequence on $R/(I + (f))$, the short exact sequence in (3.2) remains exact after tensoring with \bar{R} . It follows that $\bar{P} = \bar{I} : (\bar{f})$. Notice that $\bar{f}_1, \dots, \bar{f}_c$ is a regular sequence in \bar{R} , so we can apply Lemma 3.2 to get $\beta_0(\bar{P}) \leq (2d)^{2^{1\mathcal{B}(k,d)-1}}$ for $1\mathcal{B}(k, d) \geq 4$, or $1\mathcal{B}(k, d) = 3$ and $d \geq 3$.

If $1\mathcal{B}(k, d) = 2$, it is clear that $\beta_0(P) = \beta_0(P \cap S) \leq 2$. If $1\mathcal{B}(k, d) = 3$ and $d = 2$, then P is a minimal prime of an ideal generated by two quadrics. Let $e(R/P)$ denote the Hilbert-Samuel multiplicity of R/P with respect to the maximal ideal (x_1, \dots, x_N) , we have $e(R/P) \leq 4$. If either P contains a linear form or $e(R/P) = 4$, then P is a complete intersection and so

$\beta_0(P) = 2$. Otherwise P contains no linear forms and $e(R/P) = 3 = 1 + \text{ht}(P)$, so we can apply [13, Theorem 4.2] to see that R/P is Cohen-Macaulay and hence $\beta_0(P) \leq e(R/P) = 3$. \square

Remark 3.4. By using known results, there is a quick and simple way to obtain a worse upper bound than the one derived from Lemma 3.3. The outline is the following. First one uses the bound on the degrees of the generators of initial ideals given in [10, Corollary 3.6] and the proof of [10, Corollary 3.7] to bound the degrees of the generators of $\overline{P} = \overline{I} : (\overline{f})$. When $d \geq 2$, the bound one gets is $(d^c(cd - c))^{2^{s+1-c}} \leq (d^2(2d - 2))^{2^{\mathcal{B}(k,d)-1}} =: D$. Hence, by linear independency, the number of minimal homogeneous generators of \overline{P} is at most $\binom{\mathcal{B}(k,d)+D}{D}$.

In the proof of Lemma 3.3, notice that we could replace ${}^1\mathcal{B}(k, d)$ by the projective dimension of R modulo the minimal prime P . The same proof will give us the bound:

Theorem 3.5. *Let K be an algebraically closed field, $P \subset K[x_1, \dots, x_N]$ be a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree at most d . Then the minimal number of generators of P is bounded by*

$$\beta_0(P) \leq (2d)^{2^{pd(R/P)-1}} \leq (2d)^{2^{N-1}}.$$

Theorem 3.6. *Let S be a standard graded polynomial ring over a field \mathbb{K} , and $P \subseteq S$ a homogeneous prime ideal of height h . For every $j \geq 0$ we have that*

$$\beta_{0,j}(P) \leq h^{2^{j+1}-3}.$$

$$\begin{aligned} \sum_{j=1}^n h^{2^{j+1}-3} &= \sum_{j=1}^{n-1} h^{2^{j+1}-3} + h^{2^{n+1}-3} \\ &\leq (n-1)h^{2^n-3} + h^{2^{n+1}-3} \\ &\leq h^{2^n} h^{2^n-3} + h^{2^{n+1}-3} \\ &= 2(h^{2^{n+1}-3}). \end{aligned} \tag{3.3}$$

$$D(k, d) = 2(k^{2^{d+1}-3}).$$

Let P be a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree at most d . For the purpose of bounding the projective dimension via AH's proof, notice that we may take $D(k, d)$ to be an upper bound of $\sum_{j=1}^d \beta_{0,j}(P)$. To see why this is true, we shall briefly explain the proof in the second paragraph of section 4 AH. The goal is to show that $\text{ht}(DF)R \geq \eta + 2$ provided that the strength of a d -form F is at least ${}^3\mathcal{B}(D(\eta + 1, d - 1), d - 1) + 1$. Assume for contradiction that $\text{ht}(DF)R < \eta + 1$, then DF can be contained in a prime ideal P where P is a minimal prime of an ideal generated by a regular sequence of $\eta + 1$ or fewer forms of degree d . Then let $J \subset P$ be the ideal generated by generators of P with degree at most $d - 1$. Since DF is generated by polynomials of degree $d - 1$, we get that $DF \subset J$. This is a contradiction to theorem F in AH since the number of generators of J is bounded by $D(\eta + 1, d - 1)$. Also if characteristic of the field is 0 or $p > d$, we can assume strength of a d -form F is at least $D(\eta + 1, d - 1)$ for the proof.

We conclude this section by explaining how to compute ${}^n\mathcal{B}(\delta)$ algorithmically, where $\delta = (\delta_1, \dots, \delta_d)$ is a dimension sequence (see Theorem ??). The theoretical proof of [1, §4], which contains an inductive argument on the degree d , can be made into a recursive algorithm with the bound obtained in Lemma 3.3. However due to the computational complexity of AH proof,

the bound for large degree (4 or greater) turns out to be too large to be expressed in any meaningful function, as we will estimate the bound for $d = 2, 3$ at the end of this section to give the reader a feeling about its bad behavior.

Denote ${}^{\eta}A(i) = {}^3\mathcal{B}(D(\eta + 1, i - 1), i - 1) + 1$. Then by [1, Proposition 2.6 and Theorem A], we have ${}^{\eta}A_i(\delta) = {}^{\eta}A(i) + 3(\sum_{j=1}^d \delta_j - 1)$.

Section 4 of [1] explains how to obtain ${}^{\eta}B$ from ${}^{\eta}A$, which we will describe briefly as follows. Let V be any vector space with dimension sequence δ , if for all such V , the strength of every nonzero element of V_i is at least ${}^{\eta}A(\delta)$, then let ${}^{\eta}B(\delta) = \sum_{i=1}^d \delta_i$. Otherwise there exists a V and a degree i for which an element of V_i has an ${}^{\eta}A_i(\delta)$ -collapse. In this case set ${}^{\eta}B(\delta) = \max_{\delta'} \{{}^{\eta}B(\delta')\}$, where δ' run through all dimension sequences derived from δ by keeping δ_j unchanged for $j > i$, decreasing δ_i by 1, and increasing the δ_j 's for $j < i$ by a total of $2 \cdot {}^{\eta}A_d(\delta)$. Notice that if $d = 1$, ${}^{\eta}B((\delta_1)) = \delta_1$ trivially satisfies Theorem 2.2.

Originally when we try to comprehend the above process algorithmically, we made a mistake thinking that the worst case scenario would be δ_d decrease by 1 and δ_{d-1} increase by $2 \cdot {}^{\eta}A_d(\delta)$. However after a closer examination, we figured that the worse case scenario turns out to be much worse. In fact we need to decrease δ_i by 1 for the smallest $2 \leq i \leq d$ such that $\delta_i \neq 0$, and increase δ_{i-1} by $2 \cdot {}^{\eta}A_d(\delta)$.

One shall see that this recursive formula does not generate a meaningful bound as d gets larger. The bound for degree 4 is already an incomputable bound which looks like a nested power tower. Roughly speaking ${}^{\eta}B(0, 0, 0, n)$ behaves like a recursive formula c_n where $c_0 = 0$ and c_i is a power tower of base 7 and height c_{i-1} .

Let us estimate the bound for $d = 2, 3$ to get a closer look at the recursive formula. With this recursive formula we have ${}^{\eta}\mathcal{B}(n, d) = {}^{\eta}B(\delta)$ where $\delta = (0, \dots, 0, n)$ has d entries. For simplicity we will estimate ${}^1\mathcal{B}(n, d)$ for $d = 2, 3$. Notice that this also estimates the bound ${}^1B(\delta)$ in Theorem 2.2 since if δ is a dimension sequence with d entries whose sum equals to n , then ${}^1B(\delta) \leq {}^1\mathcal{B}(n, d)$.

Assume $d = 2$, then

$${}^{\eta}\mathcal{B}(n, 2) = {}^{\eta}B(0, n) = {}^{\eta}B(2 \cdot {}^{\eta}A_2(\delta), n - 1) = {}^{\eta}B(a_i, n - i) = {}^{\eta}B(a_n, 0) = a_n.$$

where a_i satisfies the recurrence relation $a_0 = 0$ and

$$\begin{aligned} a_i &= a_{i-1} + 2({}^{\eta}A(2) + 3(a_{i-1} + n - (i - 1) - 1)) \\ &= 7a_{i-1} + 2{}^{\eta}A(2) + 6n - 6i. \end{aligned}$$

Notice that $a_i = 7^i(n + \frac{1}{3}{}^{\eta}A(2) - \frac{7}{6}) + i - n - \frac{1}{3}{}^{\eta}A(2) + \frac{7}{6}$ is a solution of the recurrence relation. Therefore setting $i = n$ and using the value ${}^{\eta}A(2) = {}^3\mathcal{B}(D(\eta + 1, 1), 1) + 1 = \eta + 2$ we get

$$\begin{aligned} {}^{\eta}\mathcal{B}(n, 2) &= a_n = 7^n(n + \frac{1}{3}{}^{\eta}A(2) - \frac{7}{6}) - \frac{1}{3}{}^{\eta}A(2) + \frac{7}{6} \\ &\leq 7^n(n + \eta). \end{aligned}$$

In particular

$${}^1\mathcal{B}(n, 2) \leq 7^{2n}.$$

Now assume $d = 3$, then

$$\begin{aligned}
{}^1\mathcal{B}(n, 3) &= {}^1B(0, 0, n) \leq {}^1B(0, 2 \cdot {}^1A(3) + 3n, n - 1) \\
&\leq {}^1B({}^1\mathcal{B}(2 \cdot {}^1A(3) + 4n, 2), 0, n - 1) \\
&= {}^1B(b_1, 0, n - 1) \\
&\leq {}^1B(b_1, 2 \cdot {}^1A(3) + 3b_1 + 4n, n - 2) \\
&\leq {}^1B(b_2, 0, n - 2) \\
&\leq {}^1B(b_n, 0, 0), \\
&= b_n,
\end{aligned}$$

where b_n satisfies the recurrence relation $b_0 = 0$ and

$$\begin{aligned}
b_i &= {}^1\mathcal{B}(2 \cdot {}^1A(3) + 4(b_{i-1} + n), 2) \\
&\leq 7^{4 \cdot {}^1A(3) + 8(b_{i-1} + n)}.
\end{aligned}$$

By pushing up all the exponents to the very top, we get a rough estimate as follows:

$$\begin{aligned}
b_n &\leq 7^{4 \cdot {}^1A(3) + 8(b_{n-1} + n)} \\
&\leq \exp_7^n(5 \cdot {}^1A(3) + 9n),
\end{aligned}$$

where $\exp_7^n(x) = 7^{7^{\cdot^{\cdot^{\cdot^x}}}}$ with n 7's.

Now we insert the bound $D(2, 2) = 2(2^{2^3-3})$ into ${}^1A(3) = {}^3\mathcal{B}(D(2, 2), 2) + 1$ to see that

$$\begin{aligned}
{}^1\mathcal{B}(n, 3) &\leq \exp_7^n(5 \cdot {}^1A(3) + 9n) \\
&\leq \exp_7^n(7^{2^{2^3}} + 9n) \\
&\leq \exp_7^{n+3}(n + 3).
\end{aligned}$$

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