Section 3.4  Repeated Roots; Reduction of Order

Review.  Consider the linear homogeneous equation with constant coefficient

\[ ay'' + by' + cy = 0. \]

The characteristic equation is

If the discriminant \( b^2 - 4ac > 0 \), then

If the discriminant \( b^2 - 4ac < 0 \), then

In this section we study the third case when the discriminant \( b^2 - 4ac = 0 \).
Question: How to find another solution?

We consider a specific case in the next example.

Example 1. Solve the differential equation

\[ y'' + 4y' + 4y = 0. \]
The general case (repeated roots)

If the characteristic equation have two repeated roots \( r_1 = r_2 = -\frac{b}{2a} \),

Example 2. Solve the initial value problem

\[
y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}.
\]
Summary For second order linear homogeneous equations with constant coefficients

\[ ay'' + by' + cy = 0, \]

Let \( r_1 \) and \( r_2 \) be roots of characteristic equation \( ar^2 + br + c = 0 \).

- If \( r_1 \) and \( r_2 \) are real, and \( r_1 \neq r_2 \), then

- If \( r_1 \) and \( r_2 \) are real, and \( r_1 = r_2 \), then

- If \( r_1 \) and \( r_2 \) are complex conjugate \( \lambda \pm i\mu \), then

Reduction of Order

Suppose we know \( y_1(t) \) is a solution of the linear homogeneous equation (not necessarily constant coefficients):

\[ y'' + p(t)y' + q(t)y = 0. \]

To find another solution, we let
Example 3. Given that $y_1 = t$ is a solution of

$$t^2 y'' + 2ty' - 2y = 0.$$ 

find a fundamental set of solutions.