4.1 General Theory of \( n \)th Order Linear Equations

An \( n \)th order linear differential equation is

If \( p_0 \) is nowhere zero in the interval \( I \), the equation can be written as

The initial conditions are

\[
\text{Theorem (Existence and Uniqueness)} \quad \text{If} \; p_1, p_2, \cdots, p_n \text{ and } g \text{ are}
\]

The \( n \)-th order linear homogeneous equation is

If \( y_1, y_2, \cdots, y_n \) are solutions, then

The Wronskian of solutions \( y_1, y_2, \cdots, y_n \) are
**Theorem**  If the functions \( y_1, y_2, \ldots, y_n \) are solutions of the homogeneous equation and if

**Remark**  A set of solutions \( y_1, y_2, \ldots, y_n \) whose Wronskian is nonzero are called a

**Linear dependency of functions**

The functions \( f_1, f_2, \ldots, f_n \) are said to be ___________ on an interval \( I \), if

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**Example 1.** Determine whether the functions \( f_1(t) = 1, f_2(t) = t, \) and \( f_3 = t^2 \) are linearly independent or dependent on the interval \( I : -\infty < t < \infty \).

**Example 2.** Determine whether the functions \( f_1(t) = 1, f_2(t) = 2 + t, f_3 = 3 - t^2, \) and \( f_4 = 4t + t^2 \) are linearly independent or dependent on the interval \( I : -\infty < t < \infty \).
Theorem

- If the functions $y_1, y_2, \cdots, y_n$ form a fundamental set of solutions
- If $y_1, y_2, \cdots, y_n$ are linearly independent on some interval $I$

For the nonhomogeneous equation

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t),$$

the general solution is

4.2 Homogeneous Equations with Constant Coefficients

We consider the $n$th order linear homogeneous equation with constant coefficients:

if $y = e^{rt}$ is a solution, then

Real and distinct roots

If the characteristic equation has $n$ real and distinct roots, then

Example 3. Find the general solution of

$$2y''' - 4y'' - 2y' + 4y = 0.$$
Complex roots
If the characteristic equation has complex roots, then

Example 4. Find the general solution of

\[ y^{(4)} - y = 0. \]

Repeated roots
If the characteristic equation has a repeated root \( r_1 \) and multiplicity is \( s \), then

If the characteristic equation has a pair of complex roots \( \lambda + i\mu \) repeated \( s \) times, then

Example 5. Find the general solution of

\[ y^{(4)} + 2y'' + y = 0. \]