7.6 Complex Eigenvalues

In this section, we consider the system of linear homogeneous equations with constant coefficients. We focus on the case that the coefficient matrix has complex eigenvalues.

Example 1. Find the general solution

\[ x' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} x. \]
In general, we consider the system
\[ x' = Ax. \]
where \( A \) has a pair of complex conjugate eigenvalues \( r_{1,2} = \lambda \pm i\mu \).

**phase portrait**

If \( A \) is a \( 2 \times 2 \) matrix, we can visualize the solution is the \( x_1x_2 \)-plane, called phase plane, by evaluating \( Ax \) at a large number of points. More precisely, we can include some solution curves, and a plot shows sample solution curves for a given system is called a phase portrait.

- If \( r_1, r_2 \) are real and have same sign

- If \( r_1, r_2 \) are real and have opposite sign

- If \( r_{1,2} = \lambda \pm i\mu \)
Example 2. Consider the system

$$x' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} x.$$ 

Describe how the solution depends qualitatively on $\alpha$. In particular, find the critical values of $\alpha$ at which the qualitative behavior of the trajectory in the phase plane changes markedly.