MA 265 Lecture 24

Section 4.9 Rank of a Matrix (continued)

Example 1. *Let A be a 6×4 matrix.*

- *What is the largest possible value of rank (A)?*
- *Are rows of A linearly independent?*
- *Are columns of A linearly independent?*

Example 2. *Let A be a 4×5 matrix.*

- *List all possible values of the rank of A.*
- *If the rank of A is 4, what is the dimension of its column space?*
- *If the rank of A is 3, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?*

Example 3. *Let*

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & 6 & 0 \end{bmatrix}$$

Find the rank of A and the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

Theorem If A is an $m \times n$ matrix, then

Properties of Rank

Let A be an $n \times n$ matrix.

- rank(A) = n if and only if $\underline{A \text{ is row equivalent to} \qquad}$.

- A is nonsingular if and only if $\underline{\text{rank}(A) = \qquad}$.

- rank(A) = n if and only if $\underline{\det(A) \qquad}$.

- The homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\underline{\text{rank}(A) \qquad}$.

- The nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\underline{\text{rank}(A) \qquad}$.

Example 4. *Determine whether the homogeneous system has a nontrivial solution.*

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Application to Nonhomogeneous System $A\mathbf{x} = \mathbf{b}$

Theorem The linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if

Example 5. *Determine whether the nonhomogeneous system has a solution.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 4 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Remark For the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, where A is an $n \times n$ matrix,

- If $rank(A) = n$,
- If $rank(A) < n$, and $rank(A) = rank([A \mid \mathbf{b}])$,
- If $rank(A) < n$, and $rank(A) \neq rank([A \mid \mathbf{b}])$,