MA 265 Lecture 26

Section 5.3 Inner Product Spaces

Definition of Inner Product

Let V be a real vector space. An **inner product** on V is a function that assigns to each ordered pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number $(\mathbf{u}, \mathbf{v})$ satisfying

(a)

(b)

(c)

(d)

Example 1. *In $\mathbb{R}^n$, the dot product of vectors*

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is defined by

$$(\mathbf{u}, \mathbf{v}) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Show that the dot product is an inner product.

Example 2. *Compute the (standard) inner product* $(\mathbf{u}, \mathbf{v})$ *in* $\mathbb{R}_4$.

(a) $\mathbf{u} = [1\ 2\ 3\ 4]$, $\mathbf{v} = [0\ 3\ 2\ 1]$

(b) $\mathbf{u} = [1\ 2\ 3\ 4]$, $\mathbf{v} = [-4\ \ -3\ 2\ 1]$

(c) $\mathbf{u} = [\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$, $\mathbf{v} = [\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$

Definition

- A real vector space that has an inner product defined on it is called an

- In an inner product space, we define the length of a vector $\mathbf{u}$ by

In an inner product space, we have the following important inequalities:

- **Cauchy-Schwarz Inequality**:

- **Triangle Inequality**:

Proof

Example 3. *Verify Cauchy-Schwarz Inequality and triangle inequality with*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

Definition If V is an inner product space,

- we say the **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ is
- we say two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if

Definition Let V be an inner product space.

- A set S of vectors in V is called **orthogonal** if
- If, in addition,

Example 4. *Are vectors* $\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ *orthogonal? Are they orthonomal?*

Example 5. *Are vectors* $\mathbf{u_1} = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$, *orthogonal? Are they orthonomal?*

Remark Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ be an orthogonal set of nonzero vectors in an inner product space V. Then S is linearly ____________