MA 266 Lecture 10

Section 2.6 Exact Equations and Integrating Factors

In the section, we consider a special class of first order equations known as exact equations.

Example 1. *Solve the differential equation*

$$2x + y^2 + 2xyy' = 0.$$

Note: it is neither linear nor separable.

~~Observe that~~ Consider the function $\psi(x,y) = \cancel{2xy^2}\ x^2 + xy^2$

It satisfies $\frac{\partial \psi}{\partial x} = 2x + y^2 \qquad \frac{\partial \psi}{\partial y} = 2xy \cdot \cancel{\frac{dy}{dx}}$

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dx} = 0$$

Chain Rule

$$\frac{d}{dx}\left(\psi(x,y)\right) = 0$$

$$x^2 + xy^2 = c$$

Let the differential equation be

$$M(x,y) + N(x,y)y' = 0.$$

Suppose we can identify a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \qquad \frac{\partial \psi}{\partial y} = N(x,y)$$

then

$$M(x,y) + N(x,y)\,y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dx} = \frac{d}{dx}\left(\psi(x,y)\right) = 0$$

In this case, the equation is called an exact differential equation.

Question: How can we tell whether a given equation is exact?

Theorem Let M, N, M_y, and N_x be continuous on some rectangular region R. Then the equation

$$M(x,y) + N(x,y)\, y' = 0$$

is exact in R if and only if

$$\cancel{M_y(x,y)} \quad \frac{\partial}{\partial y} M = \frac{\partial}{\partial x} N$$

Question: Given an equation is exact, how to find the function ψ?

An exact equation $\frac{\partial \psi}{\partial x} = M \qquad \frac{\partial \psi}{\partial y} = N(x,y)$

Integrate the first eq. $\psi(x,y) = \int M(x,y)\, dx +$

$= Q(x,y) + h(y)$

Q is any function such that $Q_x = M$

$\psi_y = Q_y(x,y) + h'(y)$

$h'(y) = \cancel{Q_y(x,y) - N(x,y)} \; N(x,y) - Q_y(x,y)$ ← a function of y only

Example 2. *(Problem #5) Solve the differential equation*

$$\underbrace{(y\cos(x) + 2xe^y)}_{M} + \underbrace{(\sin(x) + x^2e^y - 1)}_{N}y' = 0.$$

Note that: $M_y = \cos(x) + 2xe^y$

$N_x = \cos(x) + 2xe^y$

The equation is exact: there exists a function $\psi(x,y)$ s.t.

$\psi_x = M, \quad \psi_y = N$

$\psi(x,y) = \int y\cos(x) + 2xe^y\, dx = y\cdot\sin(x) + x^2e^y + h(y)$

$\psi_y(x,y) = \sin(x) + x^2\cdot e^y + h'(y)$

$\psi_y = N$

$h'(y) = -1$

$h(y) = -y$

$\psi(x,y) = y\sin(x) + x^2e^y - y$

$y\sin(x) + x^2e^y - y = C$

$\cancel{\frac{d}{dx}(\psi(x}$