MA 266 Lecture 11

Section 2.7 Numerical Approximation: Euler's Method

In this section, we introduce a numerical method for solving the first order initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

The method is called Euler's Method or Tangent Line Method

How to use tangent lines to approximate the solution $y = \phi(t)$?

- Start with the initial point (t_0, y_0),

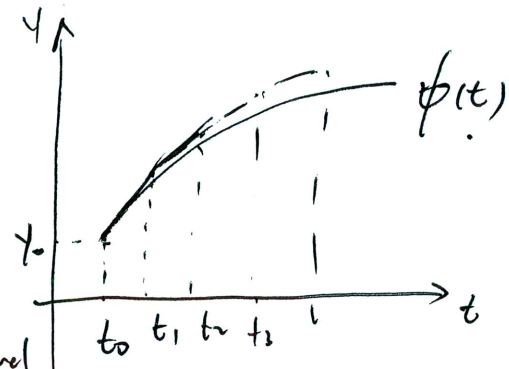
the slope at this point $f(t_0\ y_0) = m$

Tangent line $y - y_0 = m \cdot (t - t_0)$

$\cancel{y - y_0 = f}$

$y = y_0 + f(t_0, y_0) \cdot (t - t_0)$

Tangent line gives a good approximation around



- We want to continue this process with the point $(t_1, \phi(t_1))$, however,

At t_1 (close to t_0)

$$y_1 = y_0 + f(t_0, y_0)\,(t_1 - t_0)$$

However $\phi(t_1)$ is unknown. The best we can do is use y_1 instead.

point (t_1, y_1) slope $f(t_1\ y_1)$

tangent line $y = y_1 + f(t_1\ y_1) \cdot (t - t_1)$

At t_2 (close to t_1)

$$y_2 = y_1 + f(t_1\ y_1)\,(t_2 - t_1)$$

- We want to continue this process with the point $(t_1, \phi(t_1))$, however,

we don't know the value of $\phi(t_1)$. the best choice is to use y_1 instead.

the point (t_1, y_1), slope: $f(t_1, y_1)$,

tangent line: $y - y_1 = f(t_1, y_1) \cdot (t - t_1)$

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

At point t_2 (close to t_1)

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

to approximate true value $\phi(t_2)$

- The general expression for the tangent line starting at (t_n, y_n) is

$$y - y_n = f(t_n, y_n)(t - t_n)$$

$$y = y_n + f(t_n, y_n)(t - t_n)$$

The approximate value y_{n+1} at t_{n+1} in terms of t_n and y_n is

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$$

If we denote $f_n = f(t_n, y_n)$ then

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n)$$

If step size between the point $t_0, t_1, t_2, \cdots$ is uniform, say $h = t_{n+1} - t_n$ for all $n \geq 0$

$$y_{n+1} = y_n + h \cdot f_n$$

Remark. Euler's method will generate a sequence of values $y_1, y_2, \cdots$,
that approximate the values of solution $\phi(t)$ at points $t_1, t_2, t_3 \cdots$

Example 1. *Consider the initial value problem*

$$\frac{dy}{dt} = 3 - 2t - 0.5y, \qquad y(0) = 1.$$

Use Euler's method with step size $h = 0.2$ to find approximate values of solution at $t =$ $0.2, 0.4, 0.6, 0.8,$ and 1. Compare them with the corresponding values of the actual solution of the IVP.

- $t_0 = 0.$ $\quad y_0 = 1,$ $\quad \cancel{f_0(t_0, y}$ $\quad f_0 = f(t_0\ y_0) = 3 - 2\times 0 - 0.5\times 1 = 2.5$

  $y_1 = y_0 + h\cdot f_0$

  $= 1 + 0.2\cdot \cancel{f(t_0, y_0)} \times 2.5$

  $= 1.5$

- $t_1 = 0.2.$ $\quad y_1 = 1.5,$ $\quad f_1 = f(t_1\ y_1) = 3 - 2\times 0.2 - 0.5\times 1.5 = 1.85$

  $y_2 = y_1 + h\cdot f_1$

  $= 1.5 + 0.2 * 1.85$

  $= 1.87$

- $t_2 = 0.4.$ $\quad y_2 = 1.87$

  ....

~~Exact~~ True Solution: $y = \phi(t) = 14 - 4t - 13e^{-t/2}$

t	$\phi(t)$	Euler y_n
0	1	1
0.2	1.4371	1.5
0.4	1.7565	1.87
0.6	1.9694	2.1230

0.8 $\quad$ 2.0858. $\quad$ 2.2707