MA 266 Lecture 13

Section 3.1 Homogeneous Equations with Constant Coefficients

Terminologies

Starting from this section, we study the *second order differential equation* of the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

The above equation is called linear if

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

For a linear equation, we usually write it as

- (standard form)

$$y'' + p(t)y' + q(t)y = g(t)$$

- (general form)

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

An initial value problem consists of a differential equation and *a pair of* initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

A second order linear equation is called homogeneous if it has the form

$$y'' + p(t)y' + q(t)y = 0 \quad \text{or} \quad P(t)y'' + Q(t)y' + R(t)y = 0$$

Otherwise, it is called non-homogeneous

In this section, we focus on in homogeneous equations with constant coefficients, i.e.,

$$ay'' + by' + cy = 0$$

Example 1. *Solve the initial value problem*

$$\begin{cases} y'' - y = 0 \\ y(0) = 2, \quad y'(0) = -1. \end{cases}$$

For what functions, whose second-order derivative is the same as itself.

$y_1(t) = e^t$ $\qquad$ $y_2(t) = e^{-t}$

~~$y(t) = c_1 e^t + c_2 e^{-t}$~~

Any constant multiple of y_1 y_2 are solutions

$c_1 e^t$ $\qquad$ $c_2 e^{-t}$

The sum is also solution

$y(t)$ $c_1 e^t + c_2 e^{-t}$ $\qquad$ $y'(t) = c_1 e^t - c_2 e^{-t}$

$c_1 + c_2 = 2$ $\qquad$ $c_1 - c_2 = -1.$ $\quad \Rightarrow$ $c_1 = \frac{1}{2}$ $\quad c_2 = \frac{3}{2}$

$y(t) = \frac{1}{2} e^t + \frac{3}{2} e^{-t}$

In general, consider the homogeneous equation

$$ay'' + by' + cy = 0,$$

where a, b, c are given constants.

we start by looking for solutions of $y = e^{rt}$. so. $y' = r \cdot e^{r \cdot t}$

$y'' = r^2 e^{rt}$

$ar^2 e^{rt} + b \cdot r \cdot e^{r \cdot t} + c e^{rt} = 0$

$ar^2 + br + c = 0$

The equation $ar^2 + br + c = 0$ is called characteristic equation. If the characteristic equation has two *distinct real* roots r_1 and r_2, then

Then $y_1(t) = e^{r_1 t}$. $\quad y_2(t) = e^{r_2 t}$ are solutions

The general solution is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Example 2. *Solve the initial value problem*

$$\begin{cases} y'' + 5y' + 6Y = 0 \\ y(0) = 2, \quad y'(0) = 3. \end{cases}$$

C.E. $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r_1 = -3 \quad r_2 = -2.$ $\qquad y_1(t) = e^{-3t}. \quad y_2(t) = e^{-2t}$

general solution $\quad y(t) = c_1 e^{-3t} + c_2 e^{-2t}$

$y'(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t}$

I.C. $\Rightarrow$ $\begin{cases} c_1 + c_2 = 2 \\ -3c_1 - 2c_2 = 3 \end{cases}$

$\Rightarrow \begin{cases} c_1 = -7 \\ c_2 = 9 \end{cases}$

$y = -7e^{-3t} + 9e^{-2t}$

Example 3. *Consider the second order equation*

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0.$$

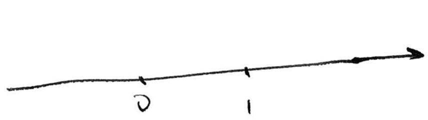
Determine the value of α, if any, for which all solutions tend to zero as $t \to \infty$. Also determine the value of α. if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

C.E. $r^2 - (2\alpha-1)r + \alpha\cdot(\alpha-1) = 0$

$(r-\alpha)(r-(\alpha-1)) = 0$

$r_1 = \alpha - 1. \quad r_2 = \alpha$

General solution $y = c_1 e^{(\alpha-1)t} + c_2 e^{\alpha t}$



If $\alpha < 0$ $\quad \lim_{t\to\infty} y = c_1 \times 0 + c_2 \times 0 = 0$

If $\alpha = 0$ $\quad \lim_{t\to\infty} y = \lim_{t\to\infty} c_1 e^{-t} + c_2 = c_2$ $\neq 0$ if $c_2 \neq 0$

If $\alpha < 1$ $\quad \lim_{t\to\infty} y = c_1 \times 0 + c_2 \times \infty = c_2 \times \infty$ $\neq 0$ if $c_2 \neq 0$

If $\alpha = 1$ $\quad \lim_{t\to\infty} y = c_1 + c_2 \times \infty$

If $\alpha > 1$ $\quad \lim y = c_1 \times \infty + c_2 \times \infty$

Hence: $\quad y \to 0$ for $\alpha < 0$

y become unbounded for $\alpha > 1$.