MA 266 Lecture 14

Section 3.2 Solutions of Linear Homogeneous Equations; Wronskian

Terminologies

$$y'' + p(t)y' + q(t)y = 0$$

In this section, we study the structure of solutions of second order linear differential equation.

Let p and q be continuous functions on an open interval I. For any function ϕ that is twice differentiable on I, we define the differential operator L by

$$L[\phi] = \phi'' + p\phi' + q\phi$$

Note that $L[\phi]$ is also a function on I. For example, let $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin(3t)$, then

$$L[\phi] = (\sin(3t))'' + t^2(\sin(3t))' + (1+t)\sin(3t)$$
$$= -9\sin(3t) + 3t^2\cos(3t) + (1+t)\sin(3t)$$

The second order homogeneous linear equation can be written as

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

associated with initial conditions:

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

where t_0 is an point in I and y_0 y_0' are real numbers.

The theoretical result of existence and uniqueness of solution is stated in the theorem.

Theorem 3.2.1 (Existence and Uniqueness) Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

where $p \cdot q \cdot g$ are continuous on an open interval I containing t_0

There there exists only one solution and the solution exists throughout I.

Example 1. *Find the longest interval in which the solution of the following initial value problem is certain to exist.*

$$\begin{cases} (t^2-3t)y''+ty'-(t+3)y=0, \\ y(1)=2, \quad y'(1)=1. \end{cases}$$

$p(t)=\frac{t}{t^2-3t}$ $q(t)=\frac{-(t+3)}{t^2-3t}$ $g(t)=0$

points of discontinuity: ~~t=~~ $t=0$ $t=3$

The longest interval containing $t=1$ is $t\in(0,3)$

Example 2. *Find the solution of the initial value problem*

$$\begin{cases} y''+p(t)y'+q(t)y=0, \\ y(0)=0, \quad y'(0)=0. \end{cases}$$

where p q are continuous in an open interval containing 0.

The function $y=0$ is a solution

By the uniqueness part. it is the only solution

Theorem (Principle of Superposition) If y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + py' + qy = 0,$$

then the linear combination

$c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1 c_2.

Proof. $L[y_1] = 0$ $L[y_2] = 0$

$$L[c_1 y_1 + c_2 y_2]$$
$$= [c_1 y_1 + c_2 y_2]'' + p(c_1 y_1 + c_2 y_2)' + q(c_1 y_1 + c_2 y_2)$$
$$= c_1 y_1'' + c_2 y_2'' + p c_1 y_1' + p c_2 y_2' + q c_1 y_1 + q c_2 y_2$$
$$= c_1(y_1'' + p y_1' + q y_1) + c_2(y_2'' + p y_2' + q y_2) = c_1 L[y_1] + c_2 L[y_2] = 0$$

Remark. Beginning with only two solutions,

we can construct infinitely solutions

Question: Are there any solution of different form?

- The initial conditions require that

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{cases} \quad \text{or} \quad \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

- The determinant of the matrix is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1 y_2' - y_1' y_2$$

- If $W \neq 0$, then there exists a unique solution $(c_1\ c_2)$

otherwise no solution

The determinant W is called the ~~Wroal~~ Wronskian of the solution y_1 and y_2.

Usually, we write it as $W(y_1, y_2)(t_0)$.

Theorem Suppose y_1 and y_2 are two solutions of

$$L[y] = y'' + py' + qy = 0,$$

Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

includes every solution of DE if and only if there is a point t_0 where the Wronskian of y_1 y_2 is nonzero

Remark.

- The theorem states that

  if and only if the Wronskian of y_1 y_2 is not every where zero, then $c_1 y_1 + c_2 y_2$ contain all solutions

- In this case, we say the expression

  $$y = c_1 y_1(t) + c_2 y_2(t)$$

  the general solution. ~~The solution y_1 y_2 form a~~

- The solutions y_1 and y_2 are said to form a ~~fund~~ fundamental set of solutions

Example 3. *Show that $y_1(t) = t^{1/2}$, and $y_2(t) = t^{-1}$ form a fundamental set of solution of*

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0.$$

First show they are solutions

$$y_1' = \frac{1}{2}t^{-\frac{1}{2}} \qquad y_1'' = -\frac{1}{4}t^{-3/2}$$

$$\ldots\ldots$$

$$y_2' = -t^{-2} \qquad y_2'' = 2t^{-3}.$$

Next: the Wronskian

$$W = y_1 y_2' - y_1' y_2$$

$$= t^{\frac{1}{2}} \cdot (-t^{-2}) - \frac{1}{2} t^{-\frac{1}{2}} \cdot t^{-1}$$

$$= -\frac{3}{2} t^{-3/2} \neq 0 \quad \text{for } t > 0$$