MA 266 Lecture 15

Section 3.3 Complex Roots of Characteristic Equation

Review. Consider the linear homogeneous equation with constant coefficient

$$ay'' + by' + cy = 0.$$

The characteristic equation is

$$ar^2 + br + c = 0$$

If the discriminant $b^2 - 4ac > 0$, then

there are two distinct real solutions

$$r = r_1 \quad \text{and} \quad r = r_2$$

In this section we study the case when the discriminant $b^2 - 4ac < 0$.

$$r_1 = \lambda \pm \mu i \qquad \lambda = \frac{-b}{2a} \qquad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

$$e^{(\lambda + \mu i)t} \qquad e^{(\lambda - \mu i)t}$$

Question: What does it mean to raise the number e to a complex power?

Euler's formula: $$e^{it} = \cos(t) + i \cdot \sin(t)$$

Euler's formula can be derived as follows

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$e^{it} = 1 + it + \frac{(it)^2}{2} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!}$$

$$\cancel{= 1 + it - \frac{t^2}{2} -}$$

$$= \cos(t) + i \cdot \sin(t)$$

$$= 1 + it + \frac{-t^2}{2} + \frac{-it^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$= \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} \cdots\right)$$

Remarks. The following identities are immediate results from Euler's formula

- $e^{-it} = \cos(t) - i\sin(t)$
- $e^{i\mu t} = \cos(\mu t) + i\sin(\mu t)$
- $e^{(\lambda+i\mu)t} = e^{\lambda t}\cdot(\cos(\mu t) + i\sin(\mu t))$

Example 1. *Find the general solution of*

$$y'' + y' + 9.25y = 0.$$

Then find the solution satisfies the initial conditions

$$y(0) = 2, \qquad y'(0) = 8.$$

C.E. $r^2 + r + \frac{37}{4} = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1^2 - 37}}{2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i$

$y_1(t) = e^{(-\frac{1}{2}+3i)t} = e^{-\frac{1}{2}t}\cdot(\cos(3t) + i\sin(3t))$

$y_2(t) = e^{(-\frac{1}{2}-3i)t} = e^{-\frac{1}{2}t}(\cos(3t) - i\sin(3t))$

$W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = -6ie^{-t} \neq 0$

$y_1\ y_2$ form a fundamental set of solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$

The expression contain imaginary part. usuall we only care real part

$u(t) = \frac{1}{2}(y_1 + y_2) = e^{-\frac{1}{2}t}\cos(3t)$. $\quad v(t) = \frac{1}{2i}(y_1 - y_2) = e^{-\frac{1}{2}t}\cdot\sin(3t)$

note that u. v are solutions

and $W(u, v)(t) = 3e^{-t} \neq 0$. u. v form a fu...

$y(t) = c_1 e^{-\frac{1}{2}t}\cos(3t) + c_2 e^{-\frac{1}{2}t}\sin(3t)$

The general case (complex roots)

If the roots of characteristic equation is $\lambda \pm \mu i$

$y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$

$y_2(t) = e^{(\lambda - i\mu)t} = e^{\lambda t}(\cos(\mu t) - i\sin(\mu t))$.

$u(t) = e^{\lambda t}\cos(\mu t))$

$v(t) = e^{\lambda t}\sin(\mu t))$

$W(u, v)(t) = \mu e^{2\lambda t}$

$y = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t))$

$= e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))$.

Example 2. *Solve the initial value problem*

$$y'' + 4y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

C.E. $r^2 + 4r + 5 = 0$ $\quad r = \cancel{\ } -2 \pm i$ $\quad \lambda = -2$. $\mu = 1$

$u(t) = e^{-2t}\cos(t)$ $\quad v(t) = e^{-2t}\sin(t)$.

$y(t) = \cancel{\ } e^{-2t}(c_1\cos(t) + c_2\sin(t))$

$y'(t) = -2e^{-2t}(\qquad\qquad) + e^{-2t}(-c_1\sin(t) + c_2\cos(t))$.

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(0) = 0 \Rightarrow -2\cdot c_1 + c_2 = 0$

$\Rightarrow \begin{cases} c_1 = 1 \\ c_2 = \cancel{\ }\,2 \end{cases}$

$y(t) = e^{-2t}(\cos(t) + 2\sin(t))$