MA 266 Lecture 16

Section 3.4 Repeated Roots; Reduction of Order

Review. Consider the linear homogeneous equation with constant coefficient

$$ay'' + by' + cy = 0.$$

The characteristic equation is

$$ar^2 + br + c = 0$$

If the discriminant $b^2 - 4ac > 0$, then

two distinct real roots, r_1, r_2.

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

If the discriminant $b^2 - 4ac < 0$, then

a pair conjugate complex roots, $r_{1,2} = \lambda \pm \mu i$.

$$c_1 e^{\lambda t} \cos\mu t + c_2 e^{\lambda t} \sin\mu t$$

In this section we study the third case when the discriminant $b^2 - 4ac = 0$.

$$r_1 = r_2 = -\frac{b}{2a}.$$

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = v(t) e^{r_1 t}$$

$$y_2'(t) = v' e^{-\frac{bt}{2a}} - \frac{b}{2a} v e^{-\frac{bt}{2a}}$$

$$y_2''(t) = v'' e^{-\frac{bt}{2a}} - \frac{b}{a} v' e^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v e^{-\frac{bt}{2a}}$$

Substitute y_2, y_2', y_2'' into $ay'' + by' + cy = 0$, we have,

$$e^{-\frac{bt}{2a}} \left\{ a\left[v'' - \frac{b}{a} v' + \frac{b^2}{4a^2} v\right] + b\left[v' - \frac{b}{2a} v\right] + cv \right\} = 0$$

$$av'' - \left[\frac{b^2-4ac}{4a}\right]v = 0$$
($\frac{b^2-4ac}{4a} \to 0$)

$$v''(t) = 0 \Rightarrow v(t) = k_3 t + k_4$$

$$y = k_1 y_1 + (k_3 t + k_4) y_1 = k_1 e^{-\frac{bt}{2a}} + (k_3 t + k_4) e^{-\frac{bt}{2a}}$$

$y = c_1 e^{-\frac{bt}{2a}} + c_2 t e^{-\frac{bt}{2a}}$, is the general solution for repeated roots.

$$y_1 = e^{-\frac{bt}{2a}}, \quad y_2 = t \cdot e^{-\frac{bt}{2a}}$$

The Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-\frac{bt}{2a}} & t e^{-\frac{bt}{2a}} \\ -\frac{b}{2a} e^{-\frac{bt}{2a}} & \left(1 - \frac{bt}{2a}\right) e^{-\frac{bt}{2a}} \end{vmatrix}$$

$$= e^{-\frac{bt}{2a}} \neq 0 \text{ for all } t.$$

y_1, y_2 form a fundamental solution set.

Question: How to find another solution?

We consider a specific case in the next example.

Example 1. *Solve the differential equation*

$$y'' + 4y' + 4y = 0.$$

C.E. $r^2+4r+4=0$ $\quad r_1 = r_2 = -2$

$y_1(t) = e^{-2t}$

$c\,y_1(t)$ is ~~another~~ also solution

Need to find a solution which is not a constant multiple of $y_1(t)$

We replace c by a function of t: $v(t)$

so that $v(t)\,y_1(t)$ is another solution of

$y(t) = v(t)\cdot e^{-2t}$

$y'(t) = v'(t)\cdot e^{-2t} + (-2)\,v(t)\,e^{-2t}$

$= (v' - 2v)\,e^{-2t}$

$y'' = (v'' - 2v')\,e^{-2t} + (v' - 2v)\cdot(-2)\,e^{-2t}$

$= e^{-2t}(v'' - 4v' + 4v).$

$e^{-2t}(v'' - 4v' + 4v + 4v' - 8v + 4v) = 0$

$y = c_1 e^{-2t} + c_2 t e^{-2t}$

$e^{-2t}\, v''(t) = 0$ $\qquad v'' = 0$

$v = c_1 t + c_2$

If $c_1 = 0$ then $v = c_2 \cdot x$. Choose $c_1 = 1$, $c_2 = 0$, $v(t) = t$

$y_2(t) = t\cdot e^{-2t}$ is another solution

$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} > 0$

The general case (repeated roots)

If the characteristic equation have two repeated roots $r_1 = r_2 = -\frac{b}{2a}$,

$y_1(t) = e^{-bt/2a}$

Let $y_2(t) = v(t)e^{-bt/2a}$

$\Rightarrow v(t) = c_1 t + c_2.$

Let $c_1 = 1 \quad c_2 = 0$

$y_2 = te^{-bt/2a}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\frac{b}{2a}e^{-bt/2a} & (1-\frac{bt}{2a})e^{-bt/2a} \end{vmatrix} = e^{-bt/a} > 0$$

$y = c_1 e^{-b/2a\, t} + c_2 t e^{-bt/2a}$

Example 2. *Solve the initial value problem*

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \qquad y'(0) = \frac{1}{3}.$$

C.E: $r^2 - r + 0.25 = 0$

$r_1 = r_2 = \frac{1}{2}$

$y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$

$y' = \frac{1}{2}c_1 e^{\frac{1}{2}t} + c_2 e^{\frac{1}{2}t} + \frac{1}{2}c_2 t e^{\frac{1}{2}t}$

$c_1 = 2. \qquad c_2 = -\frac{2}{3}$

$y = 2e^{t/2} - \frac{2}{3}t e^{t/2}$

Summary For second order linear homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0,$$

Let r_1 and r_2 be roots of characteristic equation $ar^2 + br + c = 0$.

- If r_1 and r_2 are real, and $r_1 \neq r_2$, then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

- If r_1 and r_2 are real, and $r_1 = r_2$, then

$$y = c_1 e^{r_1 t} + c_2 t e^{r_2 t}$$

- If r_1 and r_2 are complex conjugate $\lambda \pm i\mu$, then

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$$

Reduction of Order

Suppose we know $y_1(t)$ is a solution of the linear homogeneous equation (not necessarily constant coefficients):

$$y'' + p(t)y' + q(t)y = 0.$$

To find another solution, we let

$$y_2(t) = v(t) \cdot y_1(t)$$

$$y_2'(t) = v' y_1 + v \cdot y_1'$$

$$y_2''(t) = v'' y_1 + v' y_1' + v' y_1' + v y_1'' = v'' y_1 + 2v' y_1' + v y_1''$$

$$v'' y_1 + 2v' y_1' + v y_1'' + p v' y_1 + p v y_1' + q v y_1 = 0$$

$$y_1 v'' + (2y_1' + p y_1) v' + (\underbrace{y_1'' + p y_1' + q y_1}_{=0}) v = 0$$

$$y_1 v'' + (2y_1' + p y_1) \cdot v' = 0$$

$$y_1 u' + (2y_1' + p y_1) \cdot u = 0 \implies \text{First order}$$

method of reduction of order

Example 3. *Given that* $y_1 = t$ *is a solution of*

$$t^2y'' + 2ty' - 2y = 0.$$

find a fundamental set of solutions.

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = 0$$

t is a solution

$y = v(t)\cdot t$ is another solution

$y' = v'\cdot t + v$

$y'' = v''\cdot t + v' + v' = v''t + 2v'$

$v''t + 2v' + \frac{2}{t}(v't + v) - 2\cdot vt = 0$

$v''t + 2v' + 2v' + \frac{2}{t}v - \frac{2}{t^2}v\cdot t = 0$

$tv'' + 4v' = 0$

$tu' + 4u = 0$

~~$u' = -\frac{4}{t}u.$~~

$u' + \frac{4}{t}u = 0$ $\quad\quad \mu(t) = e^{\int \frac{4}{t}\,dt} = t^4$

$$\frac{d}{dt}(t^4\cdot u) = 0$$

$t^4 u = c$

$u = ct^{-4}$

$v = \int u\,dt = \int ct^{-4}\,dt = \frac{c}{-3}t^{-3} + k.$

$= ct^{-3} + k$

choose $k = 0$ $\quad c = 1$

$v(t) = t^{-3}$ $\quad y_2 = t^{-2}$ $\quad W(y_1, y_2) = \frac{-3}{t^3} \neq 0$ if $t \neq 0$