MA 266 Lecture 18

Section 3.6 Variation of Parameters

In this section, we consider a more generally applicable approach to find particular solution of nonhomogeneous equation. The method is called **variation of parameters**.

We will use the following example to illustrate the idea of this method.

Example 1. *Find a particular solution of*

$$y'' + 4y = 3\csc(t).$$

Note C.E. $r^2+4r=0$ $r_{1,2} = \pm 2i$

$y_c(t) = C_1\cos(2t) + C_2\sin(2t)$.

Basic Idea: replace C_1 and C_2 by functions $u_1(t)$ and $u_2(t)$.

so that $Y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$.

is a solution to Nonhomo eq.

$Y'(t) = u_1'\cdot\cos(2t) - 2u_1\sin(2t) + u_2'\cdot\sin(2t) + 2u_2\cos(2t)$.

We require: $u_1'\cos(2t) + u_2'\sin(2t) = 0$ ..... $(*)$.

$Y'(t) = -2u_1\sin(2t) + 2u_2\cos(2t)$

$Y''(t) = -2u_1'\sin(2t) - 4u_1\cos(2t) + 2u_2'\cos(2t) - 4u_2\sin(2t)$.

$\therefore$

(continue on the next page)

since $y'' + 4y = 3\csc(t)$

$\Rightarrow \left[2u_1'\sin(2t) - 4u_1\cos(2t) + 2u_2'\cos(2t) - 4u_2\sin(2t)\right] + \left[4u_1\cos(2t) + 4u_2\sin(2t)\right] = 3\frac{1}{\sin(t)}$

$-2u_1'\sin(2t) + 2u_2'\cos(2t) = 3\cancel{\frac{\cos(t)}{\sin(t)}}\ \csc(t) \quad \cdots (**)$

$(*) \Rightarrow u_2'(t) = -\frac{\cos(2t)}{\sin(2t)}u_1'(t) = -\cot(2t)\cdot u_1'(t)$

$2\cot(2t)\cdot\cos(2t)$

$-2u_1'\sin(2t) - \cancel{\frac{2\cos(2t)}{\sin(2t)}}\, u_1'(t) = 3\,\csc(t)$

$u_1'(t) = \frac{-3}{2}\frac{\csc(t)}{(\sin(2t) + \cot(2t)\cdot\cos(2t))} = -\frac{3}{2}\csc(t)\cdot\sin(2t)$

$= -3\cos(t)$

$\int\csc(x)dx = \ln|\sin\frac{x}{2}| - \ln|\cos\frac{x}{2}|$

$u_2'(t) = -\cot(2t)\cdot(-3\cos(t))$

$= \frac{3}{2}\csc(t) - 3\sin(t)$

$u_1(t) = -3\sin(t) + C_1$

$u_2(t) = \frac{3}{2}\ln|\csc(t) - \cot(t)| + 3\cos(t) + C_2$

The general case (variation of parameters)

We consider the equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Assume the complementary solution is

$$y_c(t) = c_1y_1(t) + c_2y_2(t). \qquad Y(t) = \ldots$$

Let $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

$Y'(t) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$. Let $u_1'y_1 + u_2'y_2 = 0 \quad \cdots (*)$

$Y''(t) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$

$u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = g.$

$u_1\underbrace{[y_1'' + py_1' + qy_1]}_{=0} + u_2\underbrace{[y_2'' + py_2' + qy_2]}_{=0} + u_1'y_1' + u_2'y_2' = g.$

$u_1'y_1' + u_2'y_2' = g \quad \cdots (**)$

Solve from $(*)$ and $(**)$ for u_1', u_2' $\Rightarrow$

$u_1' = \frac{-y_2g}{y_1y_2' - y_1'y_2} = \frac{-y_2(t)g(t)}{W(y_1,y_2)(t)}$

$u_2' = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}$

$u_1 = \int\frac{-y_2(t)g(t)}{W}dt, \quad u_2 = \int\frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt.$

Ex 3: $y'' + 2y' + y = 3e^{-t}$

C.E. $r^2 + 2r + 1 = 0$ $\quad r = -1.$

$y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$ $\qquad y_1(t) = e^{-t}$

$\qquad\qquad\qquad\qquad\qquad\qquad y_2(t) = te^{-t}$

~~Assume~~ $Y(t) = u_1(t) \cdot e^{-t} + u_2(t) t e^{-t}$ is a particular

solution. ~~then~~

$W(y_1, y_2) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - te^{-2t} + te^{-2t}$

$= e^{-2t}$

$u_1' = \dfrac{-3e^{-t} \cdot te^{-t}}{e^{-2t}} = -3t.$

$u_2' = \dfrac{e^{-t} \cdot 3e^{-t}}{e^{-2t}} = 3.$

$u_1 = -\frac{3}{2} t^2$

$u_2 = 3t$

$Y(t) = -\frac{3}{2} t^2 e^{-t} + 3t^2 \cdot e^{-t} = \frac{3}{2} t^2 e^{-t}$

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{3}{2} t^2 e^{-t}$