MA 266 Lecture 17

Section 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

We consider the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t). \qquad *$$

The corresponding homogeneous equation is

$$L[y] = y'' + p(t)y' + q(t)y = 0 \qquad **$$

The structure of the solution is summarized in the following theorem

Theorem If Y_1 and Y_2 are two solutions of nonhomogeneous equation, then
$Y_1 - Y_2$ is a solution to the corresponding homogeneous equation.

To see this, $L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$.

Theorem The general solution of nonhomogeneous equation can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1 and y_2 are a fundamental set of solution of $**$
and Y is some ~~specific~~ specific solution of $*$

Remark. To solve a nonhomogeneous equation, we do three things:

1. Find general solution $c_1 y_1 + c_2 y_2$ of $**$
2. Find a solution $Y(t)$ of $*$
3. Form

$$y = y_c(t) + Y(t) = c_1 y_1 + c_2 y_2 + Y(t)$$

Method of Undetermined Coefficient

is usually limited to when p and q are constant, and $g(t)$ is a polynomial, exponential, sine or cosine function.

Method of Undetermined Coefficients

There are generally two methods to find a particular solution $Y(t)$:

1. Method of Undertmend Coeff. and 2. Method of variation of parameters.

There are two steps for Method of undetermined coefficients

- Make an initial assumption about the form of $Y(t)$. with coefficient underfermined.
- Substitute the assumed expression into the equation. ~~inoro~~ to determd the weffrcients

1. $g(t)$ is an exponential function

Example 1. *Find a particular solution of*

$$y'' - 3y' - 4y = 3e^{2t}.$$

Assume $Y(t) = Ae^{2t}$

exponential functo reproduce itself through differentiatn

$Y'(t) = 2Ae^{2t}$

$Y''(t) = 4Ae^{2t}$

$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$

$-6A = 3$

$A = -\frac{1}{2}$.

$Y(t) = -\frac{1}{2}e^{2t}$

2. $g(t)$ **is a sine or cosine function**

Example 2. *Find a particular solution of*

$$y'' - 3y' - 4y = 2\sin(t).$$

First try:

Assume $Y = A\sin(t)$

$Y' = A\cos(t) \qquad Y'' = -A\sin(t)$

$-A\sin(t) - 3A\cos(t) - 4A\sin(t) = 2\sin(t)$

$-5A\sin(t) - 3A\cos(t) = 2\sin(t)$

$\begin{cases} -5A = 2 \\ -3A = 0 \end{cases} \qquad \times$

Second try:

Assume $Y = A\sin(t) + B\cos(t)$

$Y' = A\cos(t) - B\sin(t)$

$Y'' = -A\sin(t) - B\cos(t)$

$-A\sin(t) - B\cos(t) - 3A\cos(t) + 3B\sin(t) - 4A\sin(t) - 4B\cos(t) = 2\sin(t)$

$\begin{cases} -A + 3B - 4A = 2 \\ -B - 3A - 4B = 0 \end{cases} \Rightarrow \begin{cases} -5A + 3B = 2 \\ 3A + 5B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{5}{17} \\ B = \frac{3}{17} \end{cases}$

3. $g(t)$ **is the product of exponential and trig functions**

Example 3. *Find a particular solution of*

$$y'' - 3y' - 4y = -8e^t \cos(2t).$$

Assume: $Y(t) = A e^t \cos(2t) + B e^t \sin(2t)$

$Y' = (A+2B) e^t \cos(2t) + (-2A+B) e^t \sin(2t)$

$Y'' = (-3A+4B) e^t \cos(2t) + (-4A-3B) e^t \sin(2t)$

$Y(t) = \frac{10}{13} e^t \cos(2t) + \frac{2}{13} e^t \sin(2t)$

4. $g(t)$ is sum of two terms

If the right hand side $g(t)$ is the sum of two terms, $g(t) = g_1(t) + g_2(t)$

$$ay'' + by' + cy = g_1(t) \quad \cdots \quad Y_1(t)$$

$$ay'' + by' + cy = g_2(t) \quad \cdots \quad Y_2(t)$$

then $Y_1 + Y_2$ is a solution of the original non-homogeneous eqn.

Example 4. *Find a particular solution of*

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t) - 8e^t\cos(2t).$$

#1: $Y(t) = Ae^{2t} + B\sin(t) + C\cos(t) + De^t\sin(2t) + Ee^t\cos(2t)$.

#2: $Y_1(t) = Ae^{2t}$ for $y'' - 3y' - 4y = 3e^{2t}$

$Y_2(t) = B\sin(t) + C\cos(t)$ for $y'' - 3y' - 4y = 2\sin t$

$Y_3(t) = De^t\sin(2t) + Ee^t\cos(2t)$ for $y'' - 3y' + 4y = 8e^t\cos 2t$

$\vdots$

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t.$$

5. One difficulty

Example 5. *Find a particular solution of*

$$y'' - 3y' - 4y = 2e^{-t}.$$

$Y(t) = Ae^{-t}$ $Y'(t) = -Ae^{-t}$ $Y''(t) = Ae^{-t}$

$Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$

$0e^{-t} = 2e^{-t}$

why?

$r^2 - 3r - 4 = 0$

$r_1 = 4$ $r_2 = -1$

e^{4t} e^{-t}

Ae^{-t} is a solution to Homogeneous eq part of $Y_c(t)$.

$Y(t) = Ate^{-t}$

$Y'(t) = Ae^{-t} - Ate^{-t}$

$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

$= Ate^{-t} - 2Ae^{-t}$

$Ate^{-t} - 2Ae^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$

$-5Ae^{-t} = 2e^{-t}$

$A = -\frac{2}{5}$

So the general solution to the IVP is:

$y(t) = c_1e^{-t} + c_2e^{4t} - \frac{2}{5}te^{-t}$

Summary. The particular solution of $ay'' + by' + cy = g(t)$

- If $g(t) = P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$, then

  $Y(t) = t^s(A_0t^n + A_1t^{n-1} + \cdots A_n)$

- If $g(t) = P_n(t)e^{\alpha t}$, then

  $t^s(A_0t^n + A_1t^{n-1} \cdots + A_n)e^{\alpha t}$

- If $g(t) = P_n(t)e^{\alpha t}\sin(\beta t)$ or $g(t) = P_n(t)e^{\alpha t}\cos(\beta t)$, then

  $t^s[(A_0t^n + \cdots A_n)e^{\alpha t}\cdot\cos(\beta t) + (B_0t^n + \cdots B_n)e^{\alpha t}\cdot\sin(\beta t)]$

Here the number s is smallest nonnegative integer ($s = 0, 1, 2$) that will ensure that no term in $Y(t)$ is a solution of the homogeneous eq.