MA 266 Lecture 19

Section 3.7 Mechanic and Electrical Vibrations

In this section we use second order linear equations to model some physical processes.

Motion of a Mass on a Spring

Consider a mass m hanging at rest on the end of a vertical spring of original length l. The mass causes an elongation L of the spring in downward (positive) direction.

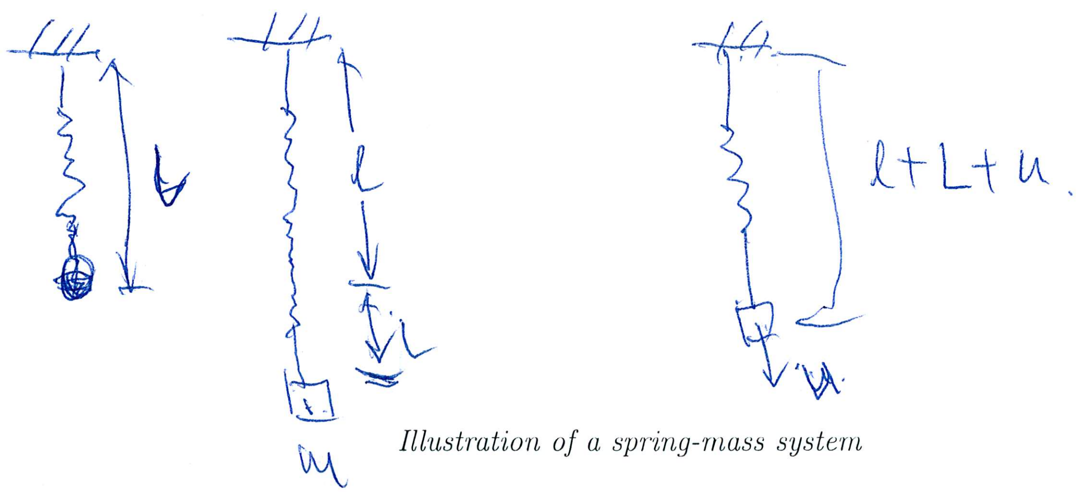

Illustration of a spring-mass system

1 Static problem

In the static status, there are two forces acting on the mass.

① . gravitational force $F_g = m\cdot g$ (+).

② . Spring force. $F_s = -k\cdot L$ (−).

$$F_g = mg = F_s = kL. \Rightarrow k = \frac{mg}{L}$$

2 Dynamic problem

We are interested in the motion of mass when

1. it is acted on by an external force,
2. is initially displaced.

Let u, measured positive direction, be the displacement of the mass from its equilibrium position at time t. By Newton's second law of motion.

$$mu''(t) = f(t) \quad \leftarrow \text{net force}$$

In the dynamic problem, there are four separate forces that must be considered:

- **gravitational force** mg : acts downwards
- **spring force** $F_s(t)$ = is proportional to $(L+u(t))$

  $$F_s(t) = -k(L+u(t))$$
- **damping force** acts in opposite direction of the motion

  ~~may arise~~ e.g.: resistance from air. internal energy dissipation

  $$F_d(t) = -\gamma u'(t)$$ proportional to velocity
- **external force**

  $F(t)$ (up. or down)

Taking account of these forces, we obtain

$$mu'' = mg + F_s(t) + F_d(t) + F(t)$$

$$= mg - k(L+u) - \gamma u' + F$$

Since $mg - kL = 0$ then

$$mu'' = -ku - \gamma u' + F(t) \qquad mu'' + \gamma u' + ku = F(t)$$

kg : mass $\qquad$ $1N = 1kg \times 9.8\ m/s^2$

lb : weights $\qquad$ slug $\qquad$ $1lb = 1 slug \times 32.2\ ft/s^2$

Example 1. *A mass weights 4 lb stretches spring 2 in. Suppose that the mass is given an additional 6 in displacement in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Formulate the initial value problem that governs the motion of the mass.*

English system.

$$m = \frac{w}{g} = \frac{4lb}{32\ ft/s^2} = \frac{1}{8}\ lb \cdot s^2/ft = \frac{1}{8}\ slug$$

Damp $\qquad \gamma = \dfrac{6lb}{3ft/s} = 2\ lb \cdot s/ft$

Spring $\qquad k = \dfrac{4lb}{2in} = \dfrac{4lb}{\frac{1}{6}ft} = \cancel{30} \cdot 24 \dfrac{lb}{ft}$

D.E. $\qquad \frac{1}{8}u'' + 2u' + 24u = 0$

$$u(0) = \frac{1}{2} \qquad u'(0) = 0$$

Undamped Free Vibrations

If there is no external force $\underline{F(t) = 0}$, and also there is no damping $\underline{\gamma = 0}$, the equation of the motion is

$$mu'' + ku = 0$$

C.E $\qquad mr^2 + k = 0$

$$r = \pm i\sqrt{k/m} = i\omega_0$$

$$u = A\cos(\omega_0 t) + \cancel{A}\sin(\omega_0 t)$$

$$u = R\cos(\omega_0 t - \delta) \Rightarrow \begin{cases} A = R\cos(\delta) \\ B = R\sin(\delta) \end{cases}$$

$R = \sqrt{A^2+B^2}$.

Period: $T = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{m}{k}}$.

Natural Frequency : $\omega_0 = \sqrt{\dfrac{k}{m}}$.

R = amplitude $\qquad \tan(\delta) = \dfrac{B}{A}$

δ = phase

Example 2. *Determine ω_0, R, and δ in order to write $u = -cos(t) + \sqrt{3}\sin(t)$ in the form $u = R\cos(\omega_0 t - \delta)$.*

$$u = -\cos(t) + \sqrt{3}\sin(t)$$

$A = -1$. $B = \sqrt{3}$

$R = 2$. $\tan\delta = \frac{B}{A} = -\sqrt{3}$.

$\cos(\delta) = -\frac{1}{2}$ $\sin(\delta) = \frac{\sqrt{3}}{2}$. $\delta = \frac{2\pi}{3}$

$$u = 2\cos\left(t - \frac{2\pi}{3}\right)$$

Remarks on undamped free vibrations

- the motion has a constant amplitude. that does not diminish with time. no energy is dissipated.
- For a given mass m and given k the system vibrates at same frequency w_0
- T increases as m increases $T = \sqrt{\frac{m}{k}}$.

  larger mass vibrates more slowly.

T decrease as k increase.

stiffer spring cause vibrate more rapidly ~~slowly~~