MA 266 Lecture 20

Section 3.7 Mechanic and Electrical Vibrations (contd)

Review For undamped free vibrations, the governing equation is

$$mu'' + ku = 0.$$

Example 1. ***(Problem 6)*** *A mass of 100 g stretches a spring 5cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?*

mass : $m = 100g = 0.1$ kg.

Spring constant: $k = \frac{mg}{L} = \frac{0.1 kg \cdot 9.8 \, m/s^2}{0.05 m} = 19.6 \, kg/s^2$

$\gamma = 0$.

no external force. $F(t) = 0$

IVP: $0.1 u'' + 19.6 u = 0$

$u(0) = 0$.

$u'(0) = 10 cm/s = 0.1 \, m/s$.

C.E: $0.1 r^2 + 19.6 = 0 \Rightarrow \boxed{r = \pm 14 i}$

$u(t) = A\cos(14t) + B\sin(14t)$

I.C: $\Rightarrow A = 0, \ B = \frac{1}{140}, \Rightarrow u(t) = \frac{1}{140}\sin(14t)$

Period $= T = \frac{2\pi}{14} = \frac{\pi}{7}$. After half period $\frac{\pi}{14}$, it returns to equilibrium position.

Damped Free Vibrations

If we include the effect of damping, the differential equation the motion becomes

$$mu'' + \gamma u' + ku = 0.$$

$$\text{C.E.}: \quad mr^2 + \gamma r + k = 0.$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

Depending the sign of $\gamma^2 - 4km$ the solution u has one of the following forms

- if $\gamma^2 - 4km > 0$ two real distinct roots. r_1, and, r_2.

  $$u = Ae^{r_1 t} + Be^{r_2 t}$$

- if $\gamma^2 - 4km = 0$ two real repeated roots, $r_1 = r_2 = -\frac{\gamma}{2m}$.

  $$u = (A + Bt)e^{-\frac{\gamma t}{2m}}$$

- if $\gamma^2 - 4km < 0$

  two complex roots.

  $$u(t) = e^{-\gamma t/2m}\left(A\cos(\mu t) + B\sin(\mu t)\right), \quad \mu = \frac{\sqrt{4km - \gamma^2}}{2m}$$

Remark

The solution u tends to zero, as $t \to \infty$ because of damping.

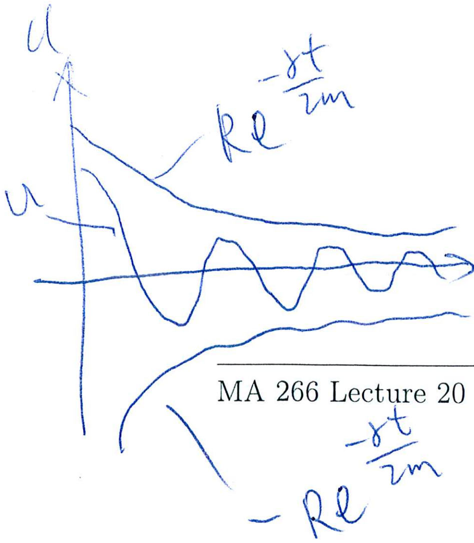
The case where $\gamma^2 - 4km < 0$ is of most interest. The solution can be written as

$$u(t) = R \cdot e^{-\gamma t/2m}\cos(\mu t - \delta)$$

$$|u(t)| \le Re^{-\gamma t/2m}$$

Displacement $u(t)$ lies between two curves

$$u = \pm Re^{-\frac{\gamma t}{2m}}$$



Example 2. ***(Problem 10)*** *A mass weighing 16lb stretches a spring 3in. The mass is attached to a viscous damper with a damping constant of 2 lb·s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/s, find its position u at any time t. Determine when the mass first returns to its equilibrium position.*

mass, $m = \frac{w}{g} = \frac{16\, lb}{32\, ft/s^2} = \frac{1}{2}\ lb\, s^2/ft$.

damping, $\gamma = 2\ lb \cdot s/ft$.

Spring constant, $k = mg/L = \frac{16\, lb}{3\, in} = \frac{16\, lb}{3 \cdot \frac{1}{12}\, ft} = 64\ lb/ft$

ODE: $\frac{1}{2}u'' + 2u' + 64u = 0$

I.C.: $u(0) = 0$, $u'(0) = 3\, in/s = 3 \cdot \frac{1}{12}\, ft/s = \frac{1}{4}\, ft/s$.

C.E: $r^2 + 4r + 128 = 0$

$r = -2 \pm 2\sqrt{31}\, i$.

The general solution is:

$$u = e^{-2t}\left(A\cos(2\sqrt{31}t) + B\sin(2\sqrt{31}t)\right)$$

I.C: $\Rightarrow A = 0$, $B = \frac{1}{8\sqrt{31}}$, $\boxed{u = \frac{1}{8\sqrt{31}} e^{-2t} \sin(2\sqrt{31}t)}$

$u = 0 \Rightarrow \sin(2\sqrt{31}t) = 0 \Rightarrow 2\sqrt{31}t = n\pi$, $n = \cancel{0}, 1, 2, \ldots$

Let $n = 1$, $\boxed{t = \frac{\pi}{2\sqrt{31}}}$