MA 266 Lecture 21

Section 3.8 Forced Vibration

In this section, we consider the situation in which a periodic external force is applied to a spring-mass system.

Forced Vibration with Damping

Example 1. *Suppose that the motion of a certain spring-mass system satisfies the differential equation:*

$$u'' + u' + 1.25u = 3\cos(t), \quad (*) \quad u(0) = 2, \qquad u'(0) = 3.$$

Find the solution of this initial value problem and describe the behavior of the solution for large t.

C.E: $r^2 + r + 1.25 = 0$

$r = -0.5 \pm i$

$u_c(t) = c_1 e^{-0.5t} \cos t + c_2 e^{-0.5t} \sin t$

Assume a particular solution of Eq $(*)$

$U(t) = A\cos t + B\sin t$

$(0.25A + B)\cos t + (-A + 0.25B)\sin t = 3\cos t$

$$\begin{cases} 0.25A + B = 3 \\ -A + 0.25B = 0 \end{cases} \Rightarrow A = \frac{12}{17}, \quad B = \frac{48}{17}$$

$u = c_1 e^{-0.5t}\cos t + c_2 e^{-0.5t}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t$

$u(0) = c_1 + \frac{12}{17} = 2$

$u'(0) = -\frac{1}{2}c_1 + c_2 + \frac{48}{17} = 3$

$\Rightarrow c_1 = \frac{22}{17}, \quad c_2 = \frac{14}{17}$

$u = \frac{22}{17} e^{-0.5t}\cos t + \frac{14}{17} e^{-0.5t}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t$

In general, the equation of motion of a spring-mass system subject to an external force $F(t)$ is

$$u = c_1 u_1(t) + c_2 u_2(t) + A\cos\omega t + B\sin\omega t$$

Forced Vibration without Damping

In this case, the motion is governed by the equation

$$mu'' + ku = F_0\cos(\omega t).$$

- If $\omega \neq \omega_0$, $u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \dfrac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t$

- If $\omega = \omega_0$, $u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \dfrac{F_0}{2m\omega_0}t\sin\omega t$

Example 2. ***(Problem 10)*** *A mass weighing 8 lb stretches a spring 6 in. The mass is acted on by an external force of* $8\sin(8t)$ *lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.*

mass $m = \dfrac{8\,lb}{32\,ft/s^2} = \dfrac{1}{4}\,\dfrac{lb\,s^2}{ft}$.

Spring $k = \dfrac{8\,lb}{\frac{1}{2}\,ft} = 16\,lb/ft$

IVP. $\begin{cases} \frac{1}{4}u'' + 16u = 8\sin(8t) \\ u(0) = \frac{1}{4}\,ft, \quad u'(0) = 0 \end{cases}$

$u = c_1\cos 8t + c_2\sin 8t - \dfrac{8t\cos 8t}{2\cdot\frac{1}{4}\cdot 8}$

$= c_1\cos 8t + c_2\sin 8t - 2t\cos 8t.$

$u(0) = \frac{1}{4} = c_1 + 0 + 0 \Rightarrow c_1 = \frac{1}{4}.$

$u'(0) = 0 = 0 + 8\cdot c_2 + (-2) = 0 \Rightarrow c_2 = \frac{1}{4}.$

$$\boxed{u = \frac{1}{4}\left[\cos 8t + \sin 8t - 8t\cos 8t\right].}$$

$u' = 0 \Rightarrow -8\sin 8t + 8\cos 8t - 8\cos 8t + 64t\sin 8t = 0.$

$\sin 8t\,(8t - 1) = 0$

$t = \frac{1}{8}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \quad u' = 0$