MA 266 Lecture 22

4.1 General Theory of nth Order Linear Equations

An nth order linear differential equation is

$$P_0(t)\frac{d^ny}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

where $P_0, P_1 \cdots P_n, G$ are continuous function on some interval I.

If P_0 is nowhere zero in the interval I, the equation can be written as

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n y = g(t)$$

The initial conditions are

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \cdots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

> **Theorem (Existence and Uniqueness)** If $p_1, p_2, \cdots, p_n$ and g are continuous on the open interval I. then ~~there~~ exists exactly one solution satisfies DE and IC.

The n-th order linear homogeneous equation is

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

If $y_1, y_2, \cdots, y_n$ are solutions, then

$y(t) = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is also a solution

The Wronskian of solutions $y_1, y_2, \cdots, y_n$ are

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem If the functions $y_1, y_2, \cdots, y_n$ are solutions of the homogeneous equation and if ~~$W(y_1, \cdots y_n)$~~ the wronskian ~~to~~

$W(y_1, \cdots y_n)(t) \neq 0$ for at least one point on I. then every solution can be expressed as linear combination of $y_1 \cdots y_n$.

Remark A set of solutions $y_1, y_2, \cdots, y_n$ whose Wronskian is nonzero are called a

fundamental set of solutions

Linear dependency of functions

The functions $f_1, f_2, \cdots, f_n$ are said to be linearly dependent on an interval I, if

there exist a set of constants $k_1 \cdots k_n$ ~~such that~~, not all zero.

$$k_1 f_1(t) + k_2 f_2(t) + \cdots k_n f_n(t) = 0$$

for all $t \in I$. Otherwise, they are linearly independent.

Example 1. *Determine whether the functions $f_1(t) = 1$, $f_2(t) = t$, and $f_3 = t^2$ are linearly independent or dependent on the interval $I : -\infty < t < \infty$.*

$$k_1 \cdot 1 + k_2 \cdot t + k_3 \cdot t^2 = 0$$

linearly depen.

At $t = 0$ $\quad k_1 = 0$

At $t = 1$ $\quad k_1 + k_2 + k_3 = 0$ $\quad \Rightarrow$ $\quad k_1 = 0$, $k_2 = 0$, $k_3 = 0$

At $t = -1$ $\quad k_1 - k_2 + k_3 = 0$

Example 2. *Determine whether the functions $f_1(t) = 1$, $f_2(t) = 2 + t$, $f_3 = 3 - t^2$, and $f_4 = 4t + t^2$ are linearly independent or dependent on the interval $I : -\infty < t < \infty$.*

$$k_1 + k_2(2+t) + k_3(3-t^2) + k_4(4t+t^2) = 0$$

$$\underbrace{(k_1 + 2k_2 + 3k_3)}_{=0} + \underbrace{(k_2 + 4k_4)}_{=0} t + \underbrace{(-k_3 + k_4)}_{=0} t^2 = 0$$

infinitely many

$k_4 = 1$ $\quad k_3 = 1$. $\quad k_2 = -4$ $\quad k_1 = 5$

Theorem

- If the functions y_1, y_2, $\cdots$, y_n form a fundamental set of solutions

  then $y_1 \cdots y_n$ are linearly independent.

- If y_1, y_2, $\cdots$, y_n are linearly independent on some interval I

  then they form a fun.......

For the nonhomogeneous equation

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t),$$

the general solution is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t)$$

4.2 Homogeneous Equations with Constant Coefficients

We consider the nth order linear homogeneous equation with constant coefficients:

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$$

if $y = e^{rt}$ is a solution, then

$$(\underbrace{a_0 r^n + a_1 r^{n-1} + \cdots + a_n}_{\text{C.E.}}) e^{rt} = 0$$

Real and distinct roots

If the characteristic equation has n real and distinct roots, then

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$$

Example 3. *Find the general solution of*

$$2y''' - 4y'' - 2y' + 4y = 0.$$

$r = 1$

$r = -1$

$r = 2$

C.E. $2r^3 - 4r^2 - 2r + 4 = 0$

$2r^2(r-2) - 2(r-2) = 2(r^2-1)(r-2) = 0$

$y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t}$

Complex roots

If the characteristic equation has complex roots, then

$$e^{\lambda t}\cos(\mu t) \qquad e^{\lambda t}\sin(\mu t)$$

Example 4. *Find the general solution of*

$$y^{(4)} - y = 0.$$

$r^4 - 1 = 0$

$(r^2-1)(r^2+1) = 0$

$r_1 = 1,\ r_2 = -1$

$r_3 = +i \quad r_4 = -i$

$y = C_1 e^t + C_2 e^{-t} + C_3 \cos(t) + C_4 \sin(t)$

Repeated roots

If the characteristic equation has a repeated root r_1 and multiplicity is s, then

$$e^{r_1 t} \quad te^{r_1 t}, \ \ldots \ t^{s-1}e^{r_1 t}$$

If the characteristic equation has a pair of complex roots $\lambda + i\mu$ repeated s times, then

$$e^{\lambda t}\cos(\mu t), \quad te^{\lambda t}\cos(\mu t), \ \ldots \ t^{s-1}e^{\lambda t}\cos(\mu t)$$

$$e^{\lambda t}\sin(\mu t), \quad t \ \ldots \ \ldots \ t^{s-1}e^{\lambda t}\sin(\mu t)$$

Example 5. *Find the general solution of*

$$y^{(4)} + 2y'' + y = 0.$$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0$

$r = i, i, -i, -i$

$y = C_1 \cos(t) + C_2 \sin(t) + C_3 t\cos(t) + C_4 t\sin(t)$