MA 266 Lecture 24

6.1 Definition of the Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by *discontinuous or impulsive forcing terms.* The methods introduced in Chapter 3 are often rather awkward to use. In this chapter, we consider a new approaches based on Laplace transform.

Review of Improper Integrals

An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals:

$$\int_a^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)\,dt$$

where A is a positive real number

If the integral from a to A exists for each $A > a$. and if the limit exists as $A \to \infty$. then the improper intergral is said to be converge.

Otherwise, it ~~is diver~~ is said to diverge

Example 1. *Let $f(t) = e^{ct}$, where $t \geq 0$ and c is a constant.*

$$\int_0^\infty e^{ct}dt = \lim_{A\to\infty} \int_0^A e^{ct}\,dt$$

$$= \lim_{A\to\infty} \left(\frac{1}{c} e^{ct} \Big|_0^A \right) \quad \text{if } c \neq 0$$

$$= \lim_{A\to\infty} \left(\frac{e^{cA}-1}{c} \right)$$

If $c < 0$ then the improper integral converge to $\frac{-1}{c}$

If $c > 0$ - - - diverge

If $c = 0$ $f(t) = 1$.

$$\int_0^\infty 1\,dt = \infty$$ diverge

Example 2. *Let $f(t) = t^{-p}$, where p is a constant. Analyze the improper integral*

$$\int_1^\infty t^{-p} dt$$

If $p = 1$ $\quad \int_1^\infty t^{-1} dt = \lim_{A\to\infty} \int_1^A t^{-1} dt = \lim_{A\to\infty} \left(\ln(t) \Big|_1^A \right) =$

$= \lim_{A\to\infty} \left(\ln A - 0 \right) = \infty$ diverge

If $p \neq 1$ $\quad \int_1^\infty t^{-p} dt = \lim_{A\to\infty} \int_1^A t^{-p} dt = \lim_{A\to\infty} \frac{1}{1-p} t^{1-p} \Big|_1^A$

$= \lim_{A\to\infty} \frac{1}{1-p} (A^{1-p} - 1)$

$p > 1$ converge $\quad p < 1$ diverge

piecewise continuous functions

A function f is said to be piecewise continuous on an interval $\alpha < t < \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. f is continuous on each open subinterval
2. f approaches a finite limit as endpoints of each subinterval are approached from within subinterval

The integral of a piecewise continuous function f on $\alpha < t < \beta$ can be written as

$$\int_\alpha^\beta f(t)\,dt = \int_{t_0}^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \cdots + \int_{t_{n-1}}^{t_n} f(t)\,dt$$

Laplace transform

An Integral transform. is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t)\, f(t)\, dt.$$

where $K(s,t)$ is given function.

kernel of the transform

The Laplace transform of the function f, denoted by $\mathcal{L}\{f(t)\}$, is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt$$

we consider s is real number

Example 3. *Find the Laplace transform of the following functions*

$$f(t) = 1, \qquad g(t) = e^{at}.$$

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} \cdot f(t)\, dt = \int_0^{+\infty} e^{-st} \cdot 1\, dt$$

$$\left(= \lim_{A\to\infty} \frac{1}{-s} \cdot e^{-st} \Big|_0^A \right) = \lim_{A\to\infty} \frac{e^{-sA} - 1}{-s}$$

$$= \frac{1}{s} \quad \text{if } \underline{s > 0}.$$

$$\mathcal{L}\{g(t)\} = \int_0^{+\infty} e^{-st} \cdot e^{at}\, dt = \int_0^{+\infty} e^{(a-s)t}\, dt$$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_0^{+\infty} = \frac{1}{s-a} \quad \text{if } s > a$$

Example 4. *Let*

$$f(t) = \begin{cases} 1, & 0 \le t < 1; \\ k, & t = 1; \\ 0, & t > 1, \end{cases}$$

where k is a constant. Find its Laplace transform.

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st}\, dt + 0 + 0$$

$$= \frac{1}{-s} e^{-st} \Big|_0^1 = \frac{1 - e^{-s}}{s}$$

Does not depends on k.

Example 5. *Let* $f(t) = \sin(at)$, $t > 0$. *Find* $\mathcal{L}\{f(t)\}$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} \cdot \sin(at)\, dt \qquad s>0$$

$$= \lim_{A\to\infty}\left(\int_0^A e^{-st}\sin(at)\,dt\right)$$

$$= \lim_{A\to\infty}\left(\frac{-e^{-st}\cos(at)}{a}\Big|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos(at)\,dt\right)$$

$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)\,dt$$

$$= \frac{1}{a} - \frac{s}{a}\left(\frac{e^{-st}\sin(at)}{a}\Big|_0^A + \frac{s}{a}\int_0^A e^{-st}\sin(at)\,dt\right)$$

$$= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin(at)\,dt$$

$$= \frac{1}{a} - \frac{s^2}{a^2}F(s)$$

$$\left(1+\frac{s^2}{a^2}\right)F(s) = \frac{1}{a}$$

$$F(s) = \frac{a}{s^2+a^2}$$

$$e^{iat} = \cos(at) + i\sin(at)$$

$$e^{-iat} = \cos(at) - i\sin(at)$$

$$\sin(at) = \frac{e^{iat}-e^{-iat}}{2i}$$

$$F(s) = \int_0^{+\infty} e^{-st}\cdot\frac{e^{iat}-e^{-iat}}{2i}\,dt$$

$$= \frac{1}{2i}\int_0^{+\infty} e^{(-s+ia)t} - e^{(-s-ia)t}\,dt = \frac{1}{2i}\left(\frac{1}{-s+ia}e^{(-s+ia)t} - \frac{1}{-s-ia}e^{(-s-ia)t}\right.$$