MA 266 Lecture 25

6.2 Solution of Initial Value Problems

In this section, we show how the Laplace transform can be used to solve initial value problem for linear differential equations with constant coefficients.

Review

The Laplace transform $\mathcal{L}\{f(t)\}$ of the function f is

$$L\{1\} = \frac{1}{s}$$

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt. \qquad L\{e^{at}\} = \frac{1}{s-a}$$

Question: what is $\mathcal{L}\{f'(t)\}$?

$$\mathcal{L}\{f'(t)\} = \int_0^{+\infty} e^{-st} f'(t)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} f'(t)\,dt$$

$$= \lim_{A\to\infty} \left(e^{-st} f(t) \Big|_0^A - (-s) \int_0^A e^{-st} f(t)\,dt \right)$$

$$= \lim_{A\to\infty} \Big(\underbrace{e^{-sA} f(A)}_{0} - f(0) + s \int_0^A e^{-st} f(t)\,dt = \underline{sL\{f\} - f(0)}$$

$$\mathcal{L}\{f''(t)\} = sL\{f'(t)\} - f'(0)$$

$$= s\left(sL\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2 L\{f(t)\} - sf(0) - f'(0)$$

Example 1. *Use Laplace transform to solve the following initial value problem*

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Classic Method:

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$y(t) = c_1 e^{2t} + c_2 e^{-t}$

I.C. $\Rightarrow y(t) = \frac{2}{3} e^{-t} + \frac{1}{3} e^{2t}$

Laplace:

$\mathcal{L}\{ y''(t) - y'(t) - 2y \} = \mathcal{L}\{0\}$

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$

$s^2 \mathcal{L}\{y\} - s y(0) - y'(0) - s\mathcal{L}\{y\} + y(0) - 2\mathcal{L}\{y\} = 0$

$(s^2 - s - 2)\mathcal{L}\{y\} + (1-s) y(0) - y'(0) = 0$

Let $Y(s) = \mathcal{L}\{y\}$

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$$

To find $y(t)$ s.t $\mathcal{L}\{y(t)\} =$

Partial Fraction. $Y(s) = \frac{a}{s-2} + \frac{b}{s+1} = \frac{as + a + bs - 2b}{(s-2)(s+1)} = \frac{(a+b)s + (a-2b)}{(s-2)(s+1)}$

$$\begin{cases} a + b = 1 \\ a - 2b = -1 \end{cases} \Rightarrow \begin{cases} b = \frac{2}{3} \\ a = \frac{1}{3} \end{cases}$$

$$\mathcal{L}\{y(t)\} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$

For the general second order equation with constant coefficients:

$$ay'' + by' + cy = f(t).$$

Laplace: $a\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + b\left[s\,Y(s) - y(0)\right] + c\,Y(s) = F(s)$

$$(as^2 + bs + c)\,Y(s) = F(s) + as\,y(0) + a\,y'(0) + b\,y(0)$$

$$Y(s) = \frac{(as+b)}{as^2+bs+c}\,y(0) + \frac{a}{as^2+bs+c}\,y'(0) + \frac{F(s)}{as^2+bs+c}$$

Find $y(t)$ whose Laplace transform is $Y(s)$

Remarks

- $Y(s)$ is found by algebraic eq. rather than diff eq.
- The solution satisfies I.C. automatically
- Nonhomogeneous eq. is handled exactly the same
- can be applied to higher order

Question: How to determine $y(t)$ corresponding to $Y(s)$?

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	
$\sin(at)$	$\frac{a}{s^2+a^2}$	
$\cos(at)$	$\frac{s}{s^2+a^2}$	
$\sinh(at)$	$\frac{a}{s^2-a^2}$	
$\cosh(at)$	$\frac{s}{s^2-a^2}$	
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	

Table 6.2.1 (pg 321) in the text book contains more functions and their Laplace transforms.

Example 2. *Find the inverse Laplace transform*

$$F(s) = \frac{2s-3}{s^2+2s+10}$$

$$F(s) = \frac{2s-3}{(s+1)^2+3^2} = \frac{2(s+1)-5}{(s+1)^2+3^2}$$

$$= 2\frac{s+1}{(s+1)^2+3^2} - \frac{5}{3}\frac{3}{(s+1)^2+3^2}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$$

Example 3. *Solve the initial value problem using Laplace transform*

$$y'' + y = \sin(2t), \qquad y(0) = 2, \qquad y'(0) = 1.$$

$$s^2 Y(s) - s\cdot y(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)\,Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$Y(s) = \frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}$$

$$Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

$$(cs+d)(s^2+1) + (as+b)(s^2+4) = 2s^3+s^2+8s+6$$

$$(a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d) = 2s^3+s^2+8s+6$$

$$a=2 \quad c=0 \quad b=\frac{5}{3} \quad d=-\frac{2}{3}$$

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} + \frac{-2/3}{s^2+4}$$

$$y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$$

For n-th order equation, we need the Laplace transform

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s \cdot f^{(n-2)}(0) - f^{(n-1)}(0)$$

Example 4. *Solve the initial value problem using Laplace transform*

$$y^{(4)} - y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0,$$

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0$$

$$(s^4 - 1) Y(s) = s^2$$

$$Y(s) = \frac{s^2}{s^4 - 1}$$

$$Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}$$

$a = c = 0.$

$b = d = \frac{1}{2}$

$$Y(s) = \frac{1}{2} \frac{1}{s^2-1} + \frac{1}{2} \frac{1}{s^2+1}$$

$$y(t) = \frac{1}{2} \sinh(t) + \frac{1}{2} \sin(t)$$