MA 266 Lecture 26

6.3 Step Functions

The most interesting applications of the Laplace transform occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. In this section, we develop some additional properties that are useful to such problems.

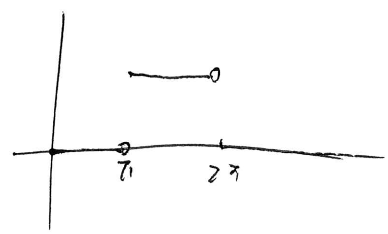
The unit step function is defined as

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

we only consider $c > 0$

Example 1. *Sketch the graph of $y = u_\pi(t) - u_{2\pi}(t)$, $t \geq 0$.*

$$y = \begin{cases} 0 & 0 < t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t < \infty \end{cases}$$



Example 2. *Express the following function in terms of $u_c(t)$.*

$$f(t) = \begin{cases} 2, & 0 \leq t < 4 \\ 5, & 4 \leq t < 7 \\ -1, & 7 \leq t < 9 \\ 1, & 9 \leq t < \infty \end{cases}$$

Start with $f_1(t) = 2$ agree with f on $[0, 4)$

At $t = 4$. a positive jump of 3 we add $3u_4(t)$ to $f_1(t)$

$f_2(t) = 2 + 3u_4(t)$

agree with f on $[0, 7)$

At $t = 7$. a negative jump of 6 Hence

$f_3(t) = 2 + 3u_4(t) - 6u_7(t)$

$f_4(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$

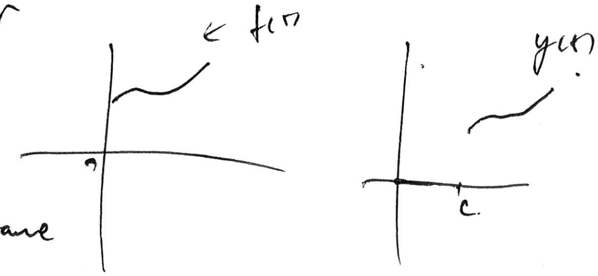
The Laplace transform of $u_c(t)$

$$\mathcal{L}\{u_c(t)\} = \int_0^{+\infty} e^{-st} u_c(t)\,dt = \int_c^{+\infty} e^{-st}\,dt = \frac{e^{-cs}}{s}, \quad s>0$$

Laplace transform of translation of a given function $f(t)$

For a given function $f(t)$ defined for $t \geq 0$, consider

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$



$g(t)$ is a translation of f of a distance c in positive direction.

$$g(t) = u_c(t)\cdot f(t-c)$$

Theorem If $F(s) = \mathcal{L}\{f(t)\}$,

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

Example 3. *Find $\mathcal{L}\{f(t)\}$ where*

$$f(t) = \begin{cases} \sin(t), & 0 \leq t < \pi/4 \\ \sin(t) + \cos(t-\pi/4), & t \geq \pi/4 \end{cases}$$

$f(t) = \sin(t) + g(t)$ where $g(t) = \begin{cases} 0 & 0 < t < \frac{\pi}{4} \\ \cos(t-\frac{\pi}{4}) & t \geq \frac{\pi}{4} \end{cases}$

$$g(t) = u_{\pi/4}\cdot\cos(t-\tfrac{\pi}{4})$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{u_{\frac{\pi}{4}}(t)\cos(t-\tfrac{\pi}{4})\} = \mathcal{L}\{\sin(t)\} + e^{-\frac{\pi}{4}s}\mathcal{L}\{\cos(t)\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} = \frac{e^{-\pi s/4}s+1}{s^2+1}$$

Example 4. *Find the inverse transform of*

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$= \mathcal{L}^{-1}\{\frac{1}{s^2}\} - \mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\}$

$= t - \mathcal{L}^{-1}\{e^{-2s} \cdot \frac{1}{s^2}\}$ or $f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2 \end{cases}$

$= t - u_2(t) f(t-2)$

Another useful property of Laplace transform:

Theorem If $F(s) = \mathcal{L}\{f(t)\}$, then

$\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$, $e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$

Example 5. *Find the inverse transform of*

$$G(s) = \frac{1}{s^2 - 4s + 5}$$

$G(s) = \frac{1}{(s-2)^2+1} = F(s-2)$ $F(s) = \frac{1}{s^2+1}$

$f(t) = \sin(t)$

$\mathcal{L}\{e^{2t} \sin(t)\} = \frac{1}{s^2 - 4s + 5}$

$g(t) = e^{2t} \sin(t)$