MA 266 Lecture 27

6.4 Differential Equations with Discontinuous Forcing Functions

In this section, we use Laplace transform to solve differential equations whose nonhomogeneous term, or forcing function, is discontinuous.

Example 1. *Solve the initial value problem*

$$y'' + y = g(t), \qquad y(0) = 0, \qquad y'(0) = 1,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & t \ge 3\pi \end{cases}$$

$$g(t) = 1 - u_{3\pi}(t)$$

Laplace: $s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{1}{s} - \frac{e^{-3\pi s}}{s}$

$$(s^2+1)Y(s) = \frac{1-e^{-3\pi s}}{s} + 1$$

$$Y(s) = \frac{s+1-e^{-3\pi s}}{s(s^2+1)}$$

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - e^{-3\pi s}\frac{1}{s(s^2+1)}$$

$$H(s) = \frac{1}{s(s^2+1)} = \frac{a}{s} + \frac{bs+c}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$h(t) = 1 - \cos(t)$$

$$y(t) = \sin(t) + 1 - \cos(t) + u_{3\pi}(t)\left(1 - \cos(t-3\pi)\right)$$

$$= \sin(t) + 1 - \cos(t) + u_{3\pi}(t)\left(1 + \cos t\right)$$

Example 2. *Solve the initial value problem*

$$y'' + 2y' + 2y = h(t), \qquad y(0) = 0, \qquad y'(0) = 1,$$

where

$$h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi, t \ge 2\pi \end{cases}$$

$h(t) = u_\pi(t) - u_{2\pi}(t)$

$$s^2 Y(s) - s y(0) - y'(0) + 2s Y(s) - 2y(0) + 2Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

$$(s^2 + 2s + 2) Y(s) = \frac{e^{-\pi s} - e^{-2\pi s} + s}{s}$$

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s \cdot (s^2 + 2s + 2)} + \frac{1}{s^2 + 2s + 2}$$

$$Y(s) = (e^{-\pi s} - e^{-2\pi s}) \left(\frac{1/2}{s} + \frac{-\frac{1}{2}s - 1}{s^2 + 2s + 2} \right) + \frac{1}{s^2 + 2s + 2}$$

$$H(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s+1}{(s+1)^2 + 1} - \frac{1}{2} \frac{1}{(s+1)^2 + 1}$$

$$h(t) = \frac{1}{2} - \frac{1}{2} e^{-t} \cdot \cos(t) - \frac{1}{2} \cdot e^{-t} \sin(t)$$

$$y(t) = u_\pi(t) \cdot h(t - \pi) - u_{2\pi}(t) h(t - 2\pi) + e^{-t} \cdot \sin(t)$$

Example 3. *Solve the initial value problem*

$$y^{(4)} - y = u_1(t) - u_2(t), \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0,$$

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = \frac{e^{-s} - e^{-2s}}{s}$$

$$Y(s) = \frac{e^{-s} - e^{-2s}}{s(s^2-1)(s^2+1)}$$

$$Y(s) = e^{-s} H(s) - e^{-2s} H(s)$$

$$H(s) = \cancel{\frac{A}{s} + \frac{bs+c}{s^2-1} + \frac{ds+e}{s^2+1}}$$

$$= \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s+1} + \frac{ds+e}{s^2+1}$$

$$= \frac{1}{4}\left(-\frac{4}{s} + \frac{1}{s-1} + \frac{1}{s+1} + \frac{2s}{s^2+1}\right).$$

$$h(t) = \frac{1}{4}\cdot\left(-4 + e^{t} + e^{-t} + 2\cdot\cos(t)\right)$$

$$y(t) = u_1(t)\, h(t-1) - u_2(t)\, h(t-2).$$