MA 266 Lecture 28

6.5 Impulse Functions

In this section, we consider the differential equation with an impulsive nature, i.e.,

$$ay'' + by' + cy = g(t),$$

where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and is otherwise zero.

The integral of $g(t)$

$$I(\tau) = \int_{-\infty}^{+\infty} g(t) = \int_{t_0-\tau}^{t_0+\tau} g(t)\, dt$$

The total impulse of force $g(t)$ over the time interval $(t_0-\tau, t_0+\tau)$

A typical $g(t)$ is defined as

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$$

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} \frac{1}{2\tau}\, dt = \frac{1}{2\tau} \cdot 2\tau = 1$$ $I(\tau)$ is independent of τ

As $\tau \to 0^+$ $I(\tau) = 1$

The delta function $\delta(t)$

unit impulse function

property:

$\delta(t) = 0$ if $t \neq 0$

$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$

An unit impulse function at $t = t_0$ is $\delta(t - t_0)$

$\delta(t - t_0) = 0$ if $t \neq t_0$

$\int_{-\infty}^{+\infty} \delta(t - t_0)\, dt =$ [illegible]

Laplace transform of $\delta(t-t_0)$

$$\mathcal{L}\{\delta(t-t_0)\} \cancel{= \int_0^{+\infty} e^{-st}\,\delta(t-t_0)\,dt}$$

$$= \lim_{\tau\to 0} \mathcal{L}\{d_\tau(t-t_0)\}$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\} = 1\ !$$

$$\mathcal{L}\{d_\tau(t-t_0)\} = \int_0^{+\infty} e^{-st}\cdot d_\tau(t-t_0)\,dt$$

$$= \int_{t_0-\tau}^{t_0+\tau} e^{-st}\cdot\frac{1}{2\tau}\,dt$$

$$= -\frac{1}{2s\tau}\cdot e^{-st}\Big|_{t_0-\tau}^{t_0+\tau}$$

$$= \frac{e^{-st_0-s\tau}-e^{-st_0+s\tau}}{-2s\tau}$$

$$= \frac{e^{-st_0}}{2s\tau}\left(e^{+s\tau}-e^{-s\tau}\right)$$

Let $\tau\to 0$

L'Hospital rule

$$= e^{-st_0}\,\frac{se^{s\tau}+se^{-s\tau}}{2s}$$

$$= e^{-st_0}$$

Let $t_0\to 0$.

$$\mathcal{L}\{\delta(t)\} = 1$$

Example 1. *Find the solution of the initial value problem*

$$2y'' + y' + 2y = \delta(t-5), \qquad y(0) = 0, \qquad y'(0) = 0.$$

$$2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = e^{-5s}$$

$$(2s^2+s+2)\,Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2}$$

$$Y(s) = \frac{e^{-5s}}{2}\,\frac{1}{s^2+\frac{1}{2}s+1}$$

$$= \frac{e^{-5s}}{2}\,\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}$$

$$\cancel{=H(s)} = \frac{e^{-5s}}{2}\cdot\frac{4}{\sqrt{15}}\,\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}$$

$$\mathcal{L}^{-1}\{H(s)\} = e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

$$y(t) = \frac{2}{\sqrt{15}}\,u_5(t)\,h(t-5)$$

Example 2. *Solve the initial value problem*

$$y'' + 2y' + 3y = \sin(t) + \delta(t-3\pi), \qquad y(0)=0, \qquad y'(0)=0.$$

$$s^2Y(s) + 2sY(s) + 3Y(s) = \frac{1}{s^2+1} + e^{-3\pi s}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s+3)} + e^{-3\pi s}\frac{1}{\cancel{(s^2+1)}(s^2+2s+3)}$$

$$= \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+2s+3} + e^{-3\pi s}\frac{1}{(s+1)^2+\sqrt{2}^2}$$

$$= \frac{-\frac{1}{4}s+\frac{1}{4}}{s^2+1} + \frac{\frac{1}{4}s+\frac{1}{4}}{s^2+2s+3} + \frac{e^{-3\pi s}}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}$$

$$= -\frac{1}{4}\cdot\frac{s}{s^2+1} + \frac{1}{4}\frac{1}{s^2+1} + \frac{1}{4}\frac{s+1}{(s+1)^2+(\sqrt{2})^2} + \frac{e^{-3\pi s}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}$$

$$y(t) = -\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t) + \frac{1}{4}e^{-t}\cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}e^{-(t-3\pi)}\sin(\sqrt{2}(t-3\pi))$$