# MA 266 Lecture 29

## 6.6 The Convolution Integral

In this section, we introduce an important tool for Laplace transform, which is known as the convolution.

**Question:** What is the inverse Laplace of $H(s)$, if $H(s) = F(s)G(s)$?

It is not $f(t) \cdot g(t)$

> If $F(s) = \mathcal{L}\{f(t)\}$, and $G(s) = \{g(t)\}$, then $H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}$, where
>
> $$h(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau = \int_0^t f(\tau) \cdot g(t-\tau)\, d\tau$$

**Remark**

- The function $h$ is not $f \cdot g$
- The function $h$ is called convolution of $f$ and $g$.

$h(t) = f * g(t)$

**The convolution $f * g$ has many properties of the ordinary multiplication**

- $f * g = g * f$
- $f * (g_1 + g_2) = f * g_1 + f * g_2$
- $f * (g * h) = (f * g) * h$
- $f * 0 = 0$

However, $f * 1 \neq f$. To see this, we let $f(t) = \cos(t)$.

$$f * 1(t) = \int_0^t \cos(t-\tau)\, d\tau = -\sin(t-\tau)\Big|_{\tau=0}^{\tau=t} = \sin(t)$$

$\neq \cos(t)$

**Example 1.** *Find the Laplace transform of*

$$f(t) = \int_0^t \sin(t-\tau)\cos(\tau)d\tau$$

$f(t) = (\sin * \cos)(t)$ $\qquad$ $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$ $\qquad$ $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

$F(s) = \frac{s}{(s^2+1)^2}$

**Example 2.** *Find the inverse transform of*

$$H(s) = \frac{1}{s^2(s^2+1)}$$

$H(s) = \frac{1}{s^2} \cdot \frac{1}{s^2+1}$

$\downarrow F(s)$ $\qquad$ $\downarrow G(s)$

$f(t) = t.$ $\quad$ $g(t) = \sin(t)$

$$\begin{aligned}
h(t) &= (f * g)(t) \\
&= \int_0^t (t-\tau)\cdot\sin(\tau)\,d\tau \\
&= \int_0^t t\sin(\tau)\,d\tau - \int_0^t \tau\sin(\tau)\,d\tau \\
&= -t\cos(\tau)\Big|_0^t - \left(-\tau\cos(\tau)\Big|_0^t + \int_0^t \cos(\tau)\,d\tau\right) \\
&= -t\cos(t) + t - \left(-t\cos(t) + \sin(t)\right) \\
&= t - \sin(t)
\end{aligned}$$

**Example 3.** *Find the solution of the initial value problem*

$$y'' + 4y = g(t), \qquad y(0) = 3, \qquad y'(0) = -1$$

$$s^2 Y(s) - 3s + 1 + 4Y(s) = G(s)$$

$$(s^2+4) \quad Y(s) = 3s - 1 + G(s)$$

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}$$

$$Y(s) = 3\frac{s}{s^2+2^2} - \frac{1}{2}\frac{2}{s^2+2^2} + \frac{1}{2}\frac{2}{s^2+4} \cdot G(s)$$

$$y(t) = 3 \cdot \cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin(2(t-\tau)) \cdot g(\tau)\, d\tau$$