MA 266 Lecture 3

Section 2.1 Linear Equations; Method of Integrating Factors

In this section, we consider the **first order linear equation**.

- The general form is $P(t)\frac{dy}{dt} + Q(t)y = G(t)$

- The standard form is $\frac{dy}{dt} + p(t)y = g(t)$

Sometimes we can solve an first order equation by integration on both sides.

Example 1. *Find the general solution of the differential equation*

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t.$$

P32

$$= \frac{d}{dt}\left[(4+t^2)y\right] = 4t$$

$$\Rightarrow (4+t^2)y = 2t^2 + c$$

$$\Rightarrow y = \frac{2t^2}{4+t^2} + \frac{c}{4+t^2}$$

Remark Most first order linear differential equations

Question: For general first order linear equations, how to solve them?

Example 2. *Find the general solution of the differential equation*

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{\frac{t}{3}}.$$

$$\underline{\mu(t)\frac{dy}{dt}} + \frac{1}{2}\underline{\mu(t)y} = \frac{1}{2}\mu e^{t/3}$$

$$\frac{d}{dt}(\mu y) = \underline{\mu\frac{dy}{dt}} + \underline{\frac{d\mu}{dt}y}$$

$$\frac{d\mu}{dt} = \frac{1}{2}\mu t$$

$$\Rightarrow \frac{d\mu/dt}{\mu} = \frac{1}{2} \Rightarrow \frac{d\ln|\mu|}{dt} = \frac{1}{2}$$

$$\Rightarrow \ln|\mu| = \frac{1}{2}t + C$$

$$\mu = ce^{t/2}$$

$$e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6}$$

$$\frac{d(e^{t/2}y)}{dt} = \frac{1}{2}e^{5t/6}$$

$$y = \frac{3}{5}e^{t/3} + ce^{-t/2}$$

In general, if the differential equation is of the form

$$\frac{dy}{dt} + ay = g(t),$$

where a is a given constant, then the integrating factor $\mu(t)$

$$\frac{d\mu}{dt} = a\mu \Rightarrow \mu = e^{at}$$

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

$$e^{at}y = \int e^{at}g\,dt + c \Rightarrow y = e^{-at}\int e^{as}g(s)\,ds + ce^{-at}$$

Example 3. *Find the general solution of*

$$y' - 2y = 3e^t.$$

and use it to determine how solution behave as $t \to \infty$.

For the more general first order linear ODE

$$\frac{dy}{dt} + p(t)y = g(t).$$

How to find an appropriate integrating factor $\mu(t)$?

$$\frac{d\mu}{dt} = p \cdot \mu \Rightarrow \ln \mu = \int p(t)\,dt + K$$

$$\mu = \exp \int p(t)\,dt.$$

$$\frac{d(\mu y)}{dt} = \mu \cdot g \Rightarrow$$

$$\mu y = \int \mu g\,dt + c$$

$$y = \frac{1}{\mu}\left(\int_{t_0}^{t} \mu g\,ds + c\right)$$

Example 4. *Solve the initial value problem*

$$ty' + 2y = 4t^2, \quad y(1) = 2.$$

$$y' + \frac{2}{t}y = 4t$$

$$\mu(t) = \exp \int \frac{2}{t}\,dt = e^{2\ln|t|} = t^2$$

$$t^2y' + 2ty = (t^2y)' = 4t^3$$

$$t^2y = t^4 + c$$

$$y = t^2 + \frac{c}{t^2}$$

$$y(1) = 2 \Rightarrow c = 1$$

$$\Rightarrow y = t^2 + \frac{1}{t^2}, \quad t > 0$$