MA 266 Lecture 30

7.1 Introduction of System of First Order Linear System

In this chapter, we study the system of first order linear differential equations.

Example 1. *Rewrite the second order equations as a system of first order equations.*

$$u'' + 0.125u' + u = 0.$$

Let $x_1 = u$ and $x_2 = u'$. Then

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.125x_2 \end{cases}$$

For an n-th order equation

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}),$$

we can transform it to a system of first order equations:

$x_1 = y$, $x_2 = y'$, $\cdots$ $x_n = y^{(n-1)}$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_{n-1}' = x_n \\ x_n' = F(t, x_1, x_2, \cdots x_n) \end{cases}$$

A more general system:

$$x_1' = F_1(t, x_1, x_2, \cdots, x_n)$$
$$x_2' = F_2(t, x_1, x_2, \cdots x_n)$$
$$\vdots$$
$$x_n' = F_n(t, x_1, x_2, \cdots x_n)$$

If $F_1, F_2, \cdots$ linear

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 \cdots p_{1n}(t)x_n + g_1(t)$$
$$x_2' =$$
$$\vdots$$
$$x_n' = p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t)$$

Theorem (Existence and Uniqueness) If the functions $p_{11}(t), p_{12}(t) \cdots p_{nn}(t)$, $g_1(t), \cdots g_n(t)$ are continuous on an open interval I, then

there exists a unique solution to the system satisfies I.C.

$x_1(0) = x_1^0$
$x_2(0) = x_2^0$
$\vdots$
$x_n(0) = x_n^0$

7.2 Review of Matrices

A matrix A with m rows and n columns can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix} \qquad A = (a_{ij})$$

- The **transpose** of A is $A^T = (a_{ji})$
- The **conjugate** of A is $\bar{A} = (\overline{a_{ij}})$
- The **adjoint** of A is $A^* = \bar{A}^T = (\overline{a_{ji}})$

For example, let

$$A = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}$$

Then

$$A^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix} \qquad \bar{A} = \begin{pmatrix} 3 & 4-3i \\ 2+i & -5-2i \end{pmatrix} \qquad A^* = \begin{pmatrix} 3 & 2+i \\ 4-3i & -5-2i \end{pmatrix}$$

special matrix: square matrix if $m=n$

vector if $n=1$

Properties of matrices

1. **Equality**: $A = B$ if and only if $a_{ij} = b_{ij}$ for i, j.
2. **Addition**: $A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ $\quad A+B = B+A.\ (A+B)+C = A+(B+C)$
3. **Scalar Multiplication**: $\alpha A = \alpha(a_{ij}) = (\alpha a_{ij})$
4. **Matrix Multiplication**: $AB \neq BA$ $\quad$ If $C = AB$, then $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Example 2. *Find AB and BA where* $(a_{ij} \cdot b_{ij})$

$$A = \begin{pmatrix} 1 & i \\ 2+i & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2i \\ -i & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1+1 & 2i+2i \\ 2+i-3i & 4i-2+6 \end{pmatrix} = \begin{pmatrix} 2 & 4i \\ 2-2i & 4+4i \end{pmatrix} \qquad BA = \begin{pmatrix} -1+4i & 7i \\ 4+i & 7 \end{pmatrix}$$

$AB \neq BA$

5. **Multiplication of Vectors**: The product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

Another definition is called **inner product**:

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \overline{y_i}$$

Remark These two products are identical if all element of $\vec{y}$ are real

$(\mathbf{x}, \mathbf{y}) = \overline{(\vec{y}, \vec{x})}$ $(\mathbf{x}, \mathbf{y}+\mathbf{z}) = (\vec{x}.\vec{y}) + (\vec{x}.\vec{z})$ $(\alpha\mathbf{x}, \mathbf{y}) = \alpha(\vec{x}, \vec{y})$ $(\mathbf{x}, \alpha\mathbf{y}) = \bar{\alpha}(\vec{x}, \vec{y})$

The inner product of $\mathbf{x}$ and itself is always nonnegative:

$$(\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} |x_i|^2$$

We define the **length** or **magnitude** or **norm** of $\mathbf{x}$

$$\|\mathbf{x}\| = (\vec{x}.\vec{x})^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

If $(\mathbf{x}, \mathbf{y}) = 0$, then the two vectors x and y are said to be orthogonal

6. **Identity**: The **identity** matrix I is defined as

$$I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix} \qquad AI = IA = A \quad \text{for any square matrix } A$$

7. **Inverse**: The matrix A is called **nonsingular** or **invertible** if

there is a matrix B such that

$AB = I$ and $BA = I$

we write $B = A^{-1}$

If a matrix don't have an inverse, it is called singular or noninvertible

Matrix Functions

We consider vectors or matrices whose elements are functions of real variable t, i.e., ~~The~~

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

derivative and integral of a matrix function are defined by

$$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt} \right) \qquad \int_a^b A(t)dt = \left(\int_a^b a_{ij}(t)\,dt \right).$$

If A, B are matrix functions, and C is a constant matrix, then

$$\frac{d}{dt}(CA) = C\frac{dA}{dt} \qquad \frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt} \qquad \frac{d}{dt}(AB) = \frac{dA}{dt}\cdot B + A\cdot\frac{dB}{dt}$$

Example 3. *Verify that the given vector satisfies the given differential equation:*

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$$

$$LHS = \vec{x}' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot 2e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$

$$RHS = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 12-4 \\ 8-4 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$