# MA 266 Lecture 31

## 7.3 Linear Dependence, Eigenvalues, Eigenvectors

### Linear Dependence

A set of $k$ vectors $\mathbf{x}^{(1)}, \cdots,$ and $\mathbf{x}^{(k)}$ is said to be linear dependent if

there exists a set of real or complex number $c_1, c_2, \cdots c_k$ at least one of which is nonzero, such that

$$c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \cdots c_k \vec{x}^{(k)} = \vec{0}$$

On the other hand, if the only set $c_1, \cdots, c_k$ satisfying the equation is

$$c_1 = c_2 = \cdots = c_k = 0$$

then $\mathbf{x}^{(1)}, \cdots,$ and $\mathbf{x}^{(k)}$ is said to be linearly independent.

For vector functions $\mathbf{x}^{(1)}(t), \cdots,$ and $\mathbf{x}^{(k)}(t)$, they are said to be **linearly dependent** on $\alpha < t < \beta$ if there exists a set of constants $c_1\ c_2 \cdots c_k$ not all zero s.t.

$$c_1 \vec{x}^{(1)}(t) + c_2 \cdots + c_k \vec{x}^{(k)}(t) = \vec{0} \quad \text{for all } t \in (\alpha, \beta)$$

**Example 1.** *Verify that the following vectors $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on the interval $0 < t < 1$*

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

$$c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = c_1 \begin{pmatrix} e^t \\ te^t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^t + c_2 \\ c_1 t e^t + c_2 t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{c_1 e^t + c_2 = 0}{c_1 t e^t + c_2 t = 0} \quad \Rightarrow \quad c_1 = 1\ c_2 = 0$$

## Eigenvalues and Eigenfunctions

The equation $A\mathbf{x} = \mathbf{y}$ can be viewed as a linear transform that maps a given vector $\mathbf{x}$ into a new vector $\mathbf{y}$. Vectors that are transformed into multiples of themselves are important in many applications. To find such vectors, we let $\mathbf{y} = \lambda\mathbf{x}$, then

$$A\vec{x} = \lambda\vec{x} \quad \text{or} \quad A\vec{x} = \lambda I\vec{x}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$\vec{x} = \vec{0}$ is a solution (trivial)

$$\underline{\det(A - \lambda I) = 0}$$

- characteristic equation of $A$
- value $\lambda$ : eigenvalue
- $\vec{x}$ : eigenvector associated with $\lambda$.

**Example 2.** *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

C.E. ~~det~~ $A - \lambda I = \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$

$\det(A - \lambda I) = 0$

$$(3-\lambda)(-2-\lambda) + 4 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0 \qquad \lambda_1 = -1, \quad \lambda_2 = 2$$

For $\lambda = -1$.

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4x_1 - x_2 = 0$

$4x_1 = x_2$

let $x_2 = 4$

$x_1 = 1$

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda = 2$

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 - x_2 = 0$

$x_1 = x_2$

choose $x_2 = 1$

$x_1 = 1$

$$\vec{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

# 7.4 Theory of System of First Order Linear Equations

The general form of a system of $n$ first order linear equations is

$$
\begin{aligned}
x_1' &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t) x_n + g_1(t) \\
\vdots &= \vdots \\
x_n' &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t) x_n + g_n(t)
\end{aligned}
$$

We can write it in matrix form

$$\vec{x}' = P(t)\,\vec{x} + \vec{g}(t)$$

The corresponding homogeneous system is

$$\vec{x}' = P(t)\,\vec{x}.$$

**Principle of Superposition** If the vector functions $\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)$ are solutions of the homogeneous system, then

~~the~~ $c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \cdots + c_n \vec{x}_n(t)$ is also a solution for any constant $c_i$

The Wronskian of these $n$ functions are

$$W(\vec{x}_1, \vec{x}_2, \cdots \vec{x}_n) = \det \left[ \vec{x}^{(1)}(t) \quad \vec{x}_2^{(2)}(t) \ \cdots \ \vec{x}^{(n)}(t) \right]$$

We say the vector functions $\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)$ are solutions form **a fundamental set of solutions** if they are linearly independent. ~~at each point~~

or the Wronskian is not zero.

In this case, each solution $\mathbf{x}(t)$ of the homogeneous system can be express as

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t)$$

If $\mathbf{x}_p(t)$ is a particular solution of the nonhomogeneous system, the general solution is

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t) + \vec{x}_p(t)$$