MA 266 Lecture 32

7.5 Homogeneous Linear Systems with Constant Coefficients

We consider the system of homogeneous linear equations with constant coefficients:

$$\mathbf{x}' = A\mathbf{x}.$$

Example 1. *Find the general solution of*

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}.$$

$$\begin{cases} x_1' = 2x_1 \\ x_2' = -3x_2 \end{cases} \Rightarrow \begin{cases} x_1 = c_1 e^{2t} \\ x_2 = c_2 e^{-3t} \end{cases}$$

$$\vec{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{-3t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

Let $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$

$\vec{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$

$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$

$W(\vec{x}^{(1)}, \vec{x}^{(2)})$

$= \det \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} = e^{-t}$

The solutions are in the form of **an exponential function multiplied by a vector.** We extend this idea to the general case. We seek solutions of the form

$$\mathbf{x} = \xi e^{rt},$$

where the exponent r and the vector ξ are to be determined.

$$r\vec{\xi} \cdot e^{rt} = A\vec{\xi} e^{rt}$$

$$(A - rI)\vec{\xi} = \vec{0}$$

If $\vec{x}$ is a solution then r is an eigenvalue and ξ is the eigenvector of A

Example 2. *Find the general solution of*

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}.$$

If $\vec{x} = \vec{\xi} e^{rt}$ is a solution then

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = 0 \qquad r_1 = -1 \quad r_2 = 3$$

For $r_1 = -1$ then $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2\xi_1 + \xi_2 = 0 \Rightarrow$

$\xi_1 = 1 \quad \xi_2 = -2$ $\qquad \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For $r_2 = 3$ then $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = -2\xi_1 + \xi_2 = 0 \Rightarrow$

$\xi_1 = 1 \quad \xi_2 = 2$ $\qquad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \qquad \vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix} = 4e^{2t}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

Example 3. *Find the solution of initial value problem*

$$\mathbf{x}' = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} \mathbf{x}. \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

~~Assume $\vec{x} = \vec{\xi} e^{rt}$ is a solution~~

$$A = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix}$$

$$\det \begin{pmatrix} 2-r & 2+i \\ -1 & -1-i-r \end{pmatrix} = (2-r)(-1-i-r) + 2+i = r^2 - r + ir - i$$

$$= (r-1)(r+i) = 0$$

$r \quad i$

$r. \quad -1$

$r_1 = 1.$

$$\begin{pmatrix} 1 & 2+i \\ -1 & -2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\xi_1 + (2+i)\,\xi_2 = 0$

$\xi_1 = +2+i$

$\xi_2 = -1$

$r_2 = -i$

$$\begin{pmatrix} 2+i & 2+i \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-\xi_1 - \xi_2 = 0$

$\xi_1 = 1$

$\xi_2 = -1$

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 2+i \\ -1 \end{pmatrix} e^{t}$$

$$\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-it}$$

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}(t))$$

$$= \begin{vmatrix} (2+i)e^{t} & e^{-it} \\ -e^{t} & -e^{-it} \end{vmatrix}$$

$$= (-2-i)e^{-it+t} + e^{-it+t}$$

$$= (-1-i)e^{-it+t}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 2+i \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-it}$$

$$X' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} X$$

Plot a phase portrait showing typical trajectories

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(-3-r)(-2-r) - 2 = r^2 + 5r + 4 = (r+1)(r+4) = 0$$

$r_1 = -1$, $r_2 = -4$. For $r=-1$,

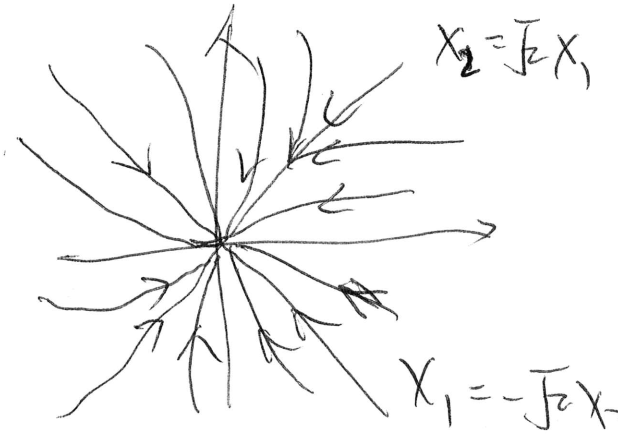
$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi_2 = \sqrt{2}\xi_1$$

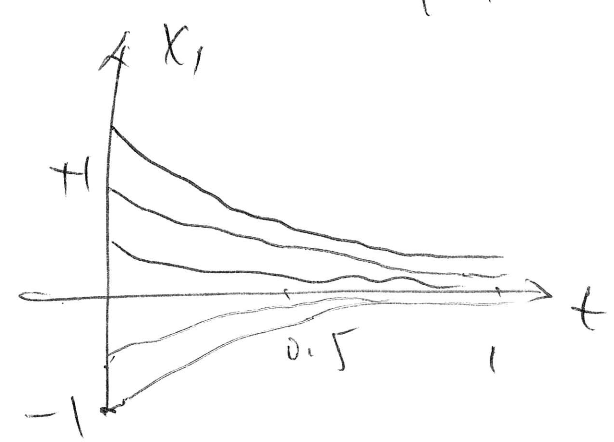
$$\xi^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

For $r=-4$, we have $\xi_1 = -\sqrt{2}\xi_2$

$$\xi^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \quad X^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad X^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

$$X = c_1 X^{(1)}(t) + c_2 X^{(2)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$





Find the general solution of

$$x' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} x$$

$$(A - rI) = \begin{vmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{vmatrix} = -r^3 + 2 - (-r - r - r) = 0$$

$$-r^3 + 3r + 2 = 0. \Rightarrow (r-2)(r+1)^2 = 0$$

$$r_1 = 2, \quad r_2 = r_3 = -1$$

For $r_1 = 2$,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $r_2 = r_3 = -1$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \quad x^{(2)}(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \quad x^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

$$x = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$