MA 266 Lecture 33

7.6 Complex Eigenvalues

In this section, we consider the system of linear homogeneous equations with constant coefficients. We focus on the case that the coefficient matrix has complex eigenvalues.

Example 1. *Find the general solution*

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}.$$

$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$ $\quad A - rI = \begin{pmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{pmatrix}$

$\det(A - rI) = r^2 + r + \frac{1}{4} + 1 = r^2 + r + \frac{5}{4} = 0$

$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i$

For $r_2 = -\frac{1}{2} - i$

$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$i\xi_1 + \xi_2 = 0$

$\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ $\quad \vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-\frac{1}{2}t - it} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-\frac{1}{2}t} (\cos(t) - i\sin(t))$

$= e^{-\frac{1}{2}t} \begin{pmatrix} \cos(t) - i\sin(t) \\ \sin(t) - i\cos(t) \end{pmatrix}$

For $r_1 = -\frac{1}{2} + i$

$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\quad \vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{1}{2}t + it} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{1}{2}t} (\cos(t) + i\sin(t))$

$= e^{-\frac{1}{2}t} \begin{pmatrix} \cos(t) + i\sin(t) \\ -\sin(t) + i\cos(t) \end{pmatrix}$

$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$

$\vec{u}(t) = \frac{\vec{x}^{(1)} + \vec{x}^{(2)}}{2} = e^{-\frac{1}{2}t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$ $\quad \vec{v}(t) = e^{-\frac{1}{2}t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$

$W(\vec{u}, \vec{v}) = e^{-t}$

$\vec{x}(t) = c_1 e^{-t/2} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$

In general, we consider the system

$$\mathbf{x}' = A\mathbf{x}.$$

where A has a pair of complex conjugate eigenvalues $r_{1,2} = \lambda \pm i\mu$.

eigenvectors: $\vec{\xi}^{(1)}$ $\vec{\xi}^{(2)}$ are also conjugate

$\overline{r_1} = r_2$ $\overline{\vec{\xi}_1} = \vec{\xi}_2$.

$$\boxed{\begin{aligned}(A - r_1 I)\vec{\xi}^{(1)} &= 0\\ (A - \bar{r}_1 I)\bar{\vec{\xi}}^{(1)} &= 0.\end{aligned}}$$

We write: $\vec{\xi}^{(1)} = \vec{a} + \vec{b}\,i$ $\quad r_1 = \lambda + i\mu$ $\quad \vec{x}^{(1)}(t)$

$\vec{\xi}^{(2)} = \vec{a} - \vec{b}\,i$ $\quad r_2 = \lambda - i\mu$ $\quad \vec{x}^{(2)}(t)$

$\vec{u}(t) = e^{\lambda t}\left(\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t)\right)$

$\boxed{\vec{v}(t) = e^{\lambda t}\left(\vec{a}\,\cancel{\cos}\sin(\mu t) + \vec{b}\,\cancel{\sin}\cos(\mu t)\right)}$ $\quad \vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$

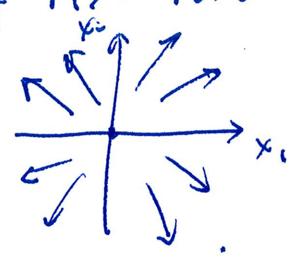
phase portrait

If A is a 2×2 matrix, we can visualize the solution is x_1x_2-plane, called **phase plane**, by evaluating $A\mathbf{x}$ at a large number of points. More precisely, we can include some solution curves, and a plot shows sample solution curves for a given system is called a **phase portrait**.

- If r_1, r_2 are real and have same sign

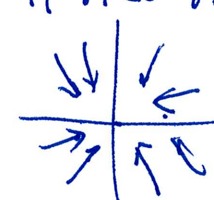
$\vec{x}(t) = c_1\vec{\xi}_1 e^{r_1 t} + c_2\vec{\xi}_2 e^{r_2 t}$.

if $r_1 > 0$ $r_2 > 0$



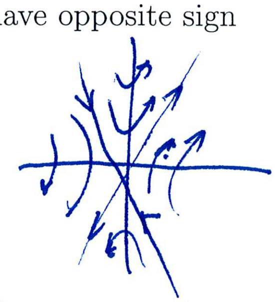
unstable node

if $r_1 < 0$ $r_2 < 0$



stable nodes

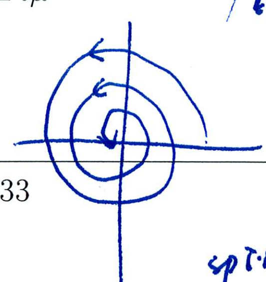
- If r_1, r_2 are real and have opposite sign



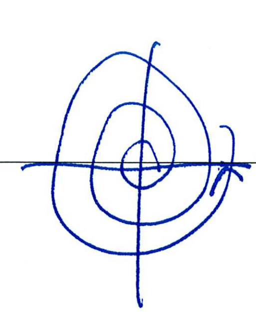
saddle point

- If $r_{1,2} = \lambda \pm i\mu$

$\lambda < 0$



spiral in



out

$\lambda > 0$ spiral point

Example 2. *Consider the system*

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}.$$

Describe how the solution depends qualitatively on α. In particular, find the critical values of α at which the qualitative behavior of the trajectory in the phase plane changes markedly.

$A = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix}$.

$\det(A - rI)$

$= \det \begin{pmatrix} \alpha - r & 2 \\ -2 & -r \end{pmatrix}$.

$= r^2 - \alpha r + 4 = 0$

$r_{1,2} = \dfrac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$

If $\alpha^2 \geq 16$. i.e. $\alpha \geq 4$ or $\alpha \leq -4$ then r is real

two critical point are $\alpha = -4$ $\alpha = 4$.

For $\alpha > 4$ ~~both~~ $r_1 > 0$ $r_2 > 0$ unstable node.

$\alpha < -4$ $r_1 < 0$ $r_2 < 0$ stable node

If ~~α~~ $-4 < \alpha < 4$ ~~$\Rightarrow$~~ eigenvalues are complex.

$-4 < \alpha < 0$ $\lambda < 0$ spiral in

$0 < \alpha < 4$ $\lambda > 0$ spiral out

$\alpha = 0$ is also a critical point

A Multiple Spring-Mass System

$$\begin{cases} m_1 \dfrac{d^2x_1}{dt^2} = (-k_1 + k_2)\, x_1 + k_2 x_2 \\ \\ m_2 \dfrac{d^2x_2}{dt^2} = k_2 x_1 - (k_2 + k_3)\, x_2 \end{cases}$$

Transform the system into a system of four first-order equations. Let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$

$$\begin{cases} y_1' = y_3 \\ y_2' = y_4 \\ m_1 y_3' = -(k_1 + k_2)\, y_1 + k_2 y_2 \\ m_2 y_4' = k_2 y_1 - (k_2 + k_3)\, y_2 \end{cases}$$

Example: $m_1 = 2$, $m_2 = 9/4$, $k_1 = 1$, $k_2 = 3$, $k_3 = 15/4$.

$$\begin{cases} y_1' = y_3 \\ y_2 = y_4 \\ y_3' = -2y_1 + \frac{3}{2} y_2 \\ y_4' = \frac{4}{3} y_1 - 3y_2 \end{cases}$$

We can write in matrix form:

$$y' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{pmatrix} y = Ay$$

We assume $y = \xi e^{rt}$, where r - eigenvalue of A, ξ the corresponding eigenvector.

The characteristic polynomial of A is:

$$r^4 + 5r^2 + 4 = (r^2+1)(r^2+4)$$

$$r_1 = i, \quad r_2 = -i, \quad r_3 = 2i, \quad r_4 = -2i.$$

$$\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \quad \xi^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}$$

$$\xi^{(1)} e^{it} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + i \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} = u^{(1)}(t) + i v^{(1)}(t)$$

$$\xi^{(3)} e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix} (\cos 2t + i \sin 2t) = \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + i \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} = u^{(2)}(t) + i v^{(2)}(t)$$

$$y = c_1 \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} + c_3 \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + c_4 \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix}$$