MA 266 Lecture 34

7.8 Repeated Eigenvalues

In this section, we consider the linear homogeneous system with constant coefficients

$$\mathbf{x}' = A\mathbf{x}$$

in which A has a repeated eigenvalue.

Example 1. *Find the eigenvalues and eigenvectors of the following matrices*

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$\det \begin{pmatrix} 2-r & 0 \\ 0 & 2-r \end{pmatrix} = (2-r)^2 = 0 \quad r_1 = r_2 = 2$

For $r_1 = 2$ $\quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Any vector is eigenvector

choose two

$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

~~For $r_2 =$~~

$\det \begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} = (r^2-4r+3+1) = (r-2)^2 = 0 \quad r_1 = r_2 = 2$

For $r_1 = 2$ $\quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \xi_1 + \xi_2 = 0 \quad \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \vec{\xi}^{(2)} = ?$

Remark. A_1 and A_2 have a repeated eigenvalue $r = 2$ with (algebraic) multiplicity 2.

- For A_1, there are two linearly independent e vectors. $\vec{x}(t) = \vec{\xi}^{(1)} e^{2t} + \vec{\xi}^{(2)} e^{2t}$
- For A_2, there is only one linearly independent eigenvector.

Need to find "another" "eigenvector"

Example 2. *Find a fundamental set of solutions of*

$$\mathbf{x}' = A_2\mathbf{x}$$

$r_1 = r_2 = 2$ $\quad \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \vec{x}^{(1)}(t) = \cancel{\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}}\ \vec{\xi}^{(1)} e^{2t}$

· Assume $\vec{x}^{(2)}(t) = \vec{\xi}\, t \cdot e^{2t}$ $\rightarrow$ For some $\vec{\xi}$ vector

$\vec{\xi} e^{2t} + \vec{\xi}\, t\, 2 \cdot e^{2t} = A \vec{\xi}\, t e^{2t}$

$\vec{\xi} \cancel{e^{2t}} + 2\vec{\xi} t - A\vec{\xi} t = 0$ for all t

$\vec{\xi} = \vec{0}$.

· Assume $\vec{x}^{(2)}(t) = \vec{\xi}\, t e^{2t} + \vec{\eta}\, e^{2t}$

$\vec{\xi} e^{2t} + 2\vec{\xi} t e^{2t} + 2\vec{\eta} e^{2t} = A\vec{\xi} t e^{2t} + A\vec{\eta} e^{2t}$

$\vec{\xi} + 2\vec{\xi} t + 2\vec{\eta} = A\vec{\xi} t + A\vec{\eta}$

$\vec{\xi} + 2\vec{\eta} = A\vec{\eta}$. $\quad A\vec{\xi} = 2\vec{\xi}$ $\quad (A - 2I)\vec{\xi} = \vec{0}$

$\vec{\xi}$ should be eigenvector of $r = 2$ $\quad \vec{\xi} = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$

$\vec{\eta}$ is solution of $(A - 2I)\vec{\eta} = \vec{\xi}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \eta_1 + \eta_2 = -1$ $\quad$ choose $\eta_1 = k$ the $\eta_2 = -1-k$

$\eta = \begin{pmatrix} k \\ -1-k \end{pmatrix}$

$\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} k \\ -1-k \end{pmatrix} e^{2t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \underline{k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}}$

$k = 0$

$\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$ $\quad W[\ \ ] = e^{-4t} \neq 0$

general case

If the matrix A has a double eigenvalue r, but there is only one eigenvector ξ, then

$\vec{x}^{(1)}(t) = \vec{\xi} e^{rt}$ $\qquad$ $(A - rI)\vec{\xi} = \vec{0}$

$\vec{x}^{(2)}(t) = \vec{\xi} t e^{rt} + \eta e^{rt}$

where $\vec{\eta}$ is determined by

$(A - rI)\vec{\eta} = \vec{\xi}$

or $(A - rI)^2 \vec{\eta} = \vec{0}$

$\vec{\eta}$: generalized eigenvector.

Fundamental Matrices are formed by arranging linearly independent solutions in columns.

Example 1. $X' = AX = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} X$

$X^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$, $X^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$

Fundamental matrix:

$$\chi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -1-t \end{pmatrix}$$

The particular fundamental matrix ϕ that satisfies $\phi(0) = I$ can be formed from $\phi(t) = \Psi(t)\Psi^{-1}(0)$

$\Psi(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $\Psi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$.

$$\phi(t) = \Psi(t)\chi^{-1}(0) = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -1-t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}.$$

which is also called the exponential matrix, $\exp(At)$