# MA 266 Lecture 35

## 7.9 Nonhomogeneous Linear Systems

In this section, we consider the nonhomogeneous system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$$

The general solution can be expressed as

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t)$$

where $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ is the general solution of the homogeneous system, and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous solution. We will introduce two methods for find a particular solution $\mathbf{v}(t)$.

### Diagonalization

If $A$ has $n$ linearly independent eigenvectors $\xi^{(1)}, \cdots, \xi^{(n)}$. Let $T = [\xi^{(1)}, \cdots, \xi^{(n)}]$. Then,

$$AT = A[\vec{\xi}^{(1)}\ \vec{\xi}^{(2)} \cdots \vec{\xi}^{(n)}] = [\lambda_1\vec{\xi}^{(1)}\ \lambda_2\vec{\xi}^{(2)} \cdots \lambda_n\vec{\xi}^{(n)}] = TD$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Hence $T^{-1}AT = D$

If $A$ is diagonalizable, then the original system can be transformed to an uncoupled system.

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

Let $\vec{x} = T\vec{y}$

$$T\vec{y}' = AT\vec{y} + \vec{g}(t)$$

multiply by $T^{-1}$

$$\vec{y}' = T^{-1}AT\vec{y} + T^{-1}\vec{g}(t)$$

$$\vec{y}' = D\vec{y} + T^{-1}\vec{g}(t)$$

For 2×2 matrix Find $T^{-1}$!

**Example 1.** *Find the general solution of*

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = A\mathbf{x} + \mathbf{g}(t)$$

$A$: eigenvalue: $r_1 = -3$. $r_2 = 1$

eigenvector: $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \cancel{T^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$T^{-1} = \sqrt{2}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $\vec{x} = T\vec{y}$ then

$$\vec{y}' = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\vec{y} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\vec{y} + \sqrt{2}\begin{bmatrix} e^{-t} - \frac{3}{2}t \\ e^{-t} + \frac{3}{2}t \end{bmatrix}$$

$$y_1' = -3y_1 + \sqrt{2}e^{-t} - \frac{3}{\sqrt{2}}t$$

$$y_2' = y_2 + \sqrt{2}e^{-t} + \frac{3}{\sqrt{2}}t$$

$$y_1 = \frac{\sqrt{2}}{2}e^{-t} - \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{6} + c_1 e^{-3t}$$

$$y_2 = \sqrt{2}te^{-t} + \frac{3\sqrt{2}}{2}t - \frac{3\sqrt{2}}{2} + c_2 e^{-t}$$

$y_1 + y_2$

$$\vec{x} = T\vec{y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\sqrt{2}}{2}e^{-t} - \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{6} + c_1e^{-3t} + \sqrt{2}te^{-t} + \frac{3\sqrt{2}}{2}t - \frac{3\sqrt{2}}{2} + c_2e^{-t} \\ -y_1 + y_2 \end{pmatrix}$$

$$\vec{x} = \frac{c_1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + \frac{c_2}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

## Undetermined Coefficients

If $\mathbf{g}(t)$ is a polynomial, exponential, or sin function, or the sum or product of these functions, we can use the method of undetermined coefficients.

**Main difference:** If $\mathbf{g}(t)$ consists of $\mathbf{u}e^{\lambda t}$, where $\lambda$ is an eigenvalue of $A$,

~~rather th~~ ~~we~~ $\vec{a} t e^{\lambda t}$ but $\vec{a} t e^{\lambda t} + \vec{b} e^{\lambda t}$

**Example 2.** *Use the method of undetermined coefficient to find a particular solution of*

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = A\mathbf{x} + \mathbf{g}(t)$$

$$\vec{g}(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\vec{g}_1} e^{-t} + \underbrace{\begin{pmatrix} 0 \\ 3 \end{pmatrix}}_{\vec{g}_2} t$$

$\lambda = -1$ is an e-value.

$$\vec{x}_p(t) = \vec{a} t e^{-t} + \vec{b} e^{-t} + \vec{c} t + \vec{d}$$

$$\vec{x}_p'(t) = \vec{a} e^{-t} - \vec{a} t e^{-t} - \vec{b} e^{-t} + \vec{c}$$

$$\vec{a} e^{-t} - \vec{a} t e^{-t} - \vec{b} e^{-t} + \vec{c} = A\vec{a} t e^{-t} + A\vec{b} e^{-t} + A\vec{c} t + A\vec{d} + \vec{g}_1 e^{-t} + \vec{g}_2 t$$

$$\begin{cases} A\vec{a} = -\vec{a} \\ A\vec{b} = \vec{a} - \vec{b} - \vec{g}_1 \\ A\vec{c} = -\vec{g}_2 \\ A\vec{d} = \vec{c} \end{cases} \Rightarrow$$

$\vec{a}$ is e-vector of $\lambda_1 = -1$ $\quad \vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \begin{matrix} -b_1 + b_2 = -1 \\ b_1 - b_2 = 1 \end{matrix}$$

$$\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$

then



$$\vec{x}_p = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Example 3 use the method of variation of parameters to find the general solution of the system:

$$x' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = Ax + g(t)$$

The general solution of the homogeneous system is:

$$x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Thus:

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \text{ is a fundamental matrix}$$

Then the solution $x = \Psi(t)\vec{u}(t)$, where $\vec{u}(t)$ satisfies

$\Psi(t)u'(t) = g(t)$ or:

$$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

Solving above equation, we obtain

$$u_1' = e^{2t} - \frac{3}{2} t e^{3t}$$

$$u_2' = 1 + \frac{3}{2} t e^{t}$$

Hence:

$$u_1(t) = \frac{1}{2} e^{2t} - \frac{1}{2} t e^{3t} + \frac{1}{6} e^{3t} + c_1$$

$$u_2(t) = t + \frac{3}{2} t e^{t} - \frac{3}{2} e^{t} + c_2$$

$$x = \Psi(t)u(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

$$+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Example 4. Use Laplace transforms to solve the system
$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = AX + g(t)$$

Take the Laplace transform of each term, obtaining
$$s\vec{X}(s) - \vec{X}(0) = A\vec{X}(s) + \vec{G}(s)$$
where $G(s)$ is the transform of $g(t)$. $G(s)$ is given b
$$G(s) = \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

We need to choose $\vec{X}(0)$. For simplicity, let $\vec{X}(0) = 0$

Then we have:
$$(sI - A)X(s) = G(s)$$
$$X(s) = (sI - A)^{-1} G(s)$$

$(sI - A)^{-1}$ is called the transfer matrix. In this example, we have:
$$(sI - A) = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}$$

and
$$(sI - A)^{-1} = \frac{1}{(s+1)(s+3)} \begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}$$

$$X(s) = (sI - A)^{-1} G(s) = \begin{pmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}$$

Expanding $X(s)$ in partial fractions, we have
$$X(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$