MA 266 Lecture 5

Section 2.2 Separable Equations (contd)

Example 1. *Consider the initial value problem*

$$y' = ty(4-y)/3, \quad y(0) = y_0 > 0.$$

(a).Determine how the behavior of the solution as t *increases depends on the initial value* y_0.
(b).Suppose that $y_0 = 0.5$. *Find the time* T *at which the solution first reaches the value* 3.98.

$$\frac{1}{y(4-y)}dy = \frac{t}{3}dt \Rightarrow \frac{1}{4}\left(\frac{1}{y} - \frac{1}{y-4}\right)dy = d\left(\frac{t^3}{6}\right)$$

$$\frac{1}{4}(\ln|y| - \ln|y-4|)dy = d\left(\frac{t^2}{6}\right)$$

$$\frac{1}{4}\ln\left|\frac{y}{y-4}\right| = \frac{t^2}{6} + C$$

$$y(0) = y_0 \Rightarrow C_0 = \frac{1}{4}\ln\left|\frac{y_0}{y_0-4}\right| - \frac{0}{6}$$

$$\Rightarrow \frac{1}{4}\ln\left|\frac{y(y_0-4)}{y_0(y-4)}\right| - \frac{t^2}{6} = 0$$

$t \to \infty \quad y \to 4.$

$y_0 = 0.5, \quad y = 3.98$

$$\Rightarrow \frac{1}{4}\ln\left|\frac{3.98\cdot(-3.5)}{0.5\cdot(-0.02)}\right| = \frac{T^2}{6}$$

$$T = \sqrt[2]{\frac{3}{2}\ln(1393)}$$

Homogeneous Equation

Consider the differential equation $\frac{dy}{dx} = f(x, y)$. If the right hand side

then the equation is said to be ________________. Such equation can be transformed into ________________________ by a change of variable.

Example 2. *Solve the differential equation*

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

Homogeneous Equation

Consider the differential equation $\frac{dy}{dx} = f(x, y)$. If the right hand side can be expressed as a function of y/x only then the equation is said to be homogeneous. Such equation can be transformed into separable equation by a change of variable.

Example 2. *Solve the differential equation*

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

$$\frac{dy}{dx} = \frac{y/x - 4}{1 - y/x}$$

Let $v = y/x$, $\frac{dy}{dx} = \frac{d(xv)}{dx} = \frac{dv}{dx} \cdot x + v$

$\hookrightarrow y = xv$

$$\frac{dv}{dx} \cdot x + v = \frac{v - 4}{1 - v}$$

$$\frac{dv}{dx} = \left(\frac{v^2 - 4}{1 - v}\right) \frac{1}{x}$$

$$\frac{(1 - v)}{(v^2 - 4)} dv = d \ln|x|$$

$$\left[-\frac{1}{v+2} - \frac{1}{4}\left(\frac{1}{v+2} - \frac{1}{v-2}\right)\right] dv = d \ln|x|$$

$$-\left[\frac{3}{4}\frac{1}{(v+2)} + \frac{1}{4}\frac{1}{(v-2)}\right] dv = d \ln|x|$$

$$-\frac{3}{4} \ln|v+2| - \frac{1}{4} \ln|v-2| = \ln|x|$$

$$\left[(v+2)^3 (v-2)\right]^{-\frac{1}{4}}$$

$$\sqrt[4]{\cancel{\frac{(v-2)}{(v+2)^5}}} = |x|$$

$$\left|\cancel{\frac{v-2}{(v+2)^5}}\right| = |x|^4$$

$$\left|\cancel{\frac{y/x - 2}{(y/x + 2)^5}}\right| = |x|^4$$

$$\Rightarrow \left|\cancel{\frac{y x^4 - 2x^5}{(y + 2x)^5}}\right| = x^4$$