MA 266 Lecture 6

Section 2.3 Modeling with First Order Equations

In this section, we consider several mathematical models using first order differential equations.

Steps in the process of mathematical modeling:

1. **Construction of the Model** translate physical situation into mathematical models
   state the physical principles governing the processes
2. **Analysis of the Model**
   solving differential eqns.
3. **Comparison with Experiment or Observation**
   interpret the solution
   check the solution physically reasonable.
   compare with experimental data.

Example 1. *(Mixing)*

At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gallons of water. Assume that water containing $1/4$ lb of salt/gal is entering the tank at a rate of r gal/min and that the well-stirred mixture is draining from the tank at the same rate. *(continue next page)*

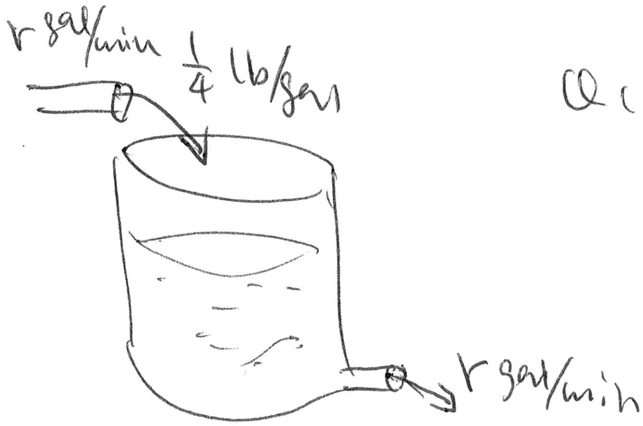
An illustration of water tank in Example 1

Answer the following questions.

1. Set up the initial value problem that describes this flow process.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= \frac{r}{4} - \frac{rQ}{100}$$



$$Q(0) = Q_0$$

2. Find the amount of salt $Q(t)$ in the tank at any time and find the limiting amount Q_L that is present after a very long time.

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}$$

Integrating factor $e^{rt/100}$

$$Q(t) = 25 + ce^{-rt/100}$$

$Q(0) = Q_0 \Rightarrow c = Q_0 - 25, \Rightarrow Q(t) = 25 + (Q_0 - 25)e^{-rt/100}$

$t \to \infty, \; Q(t) \to 25$

(Example 1 continues next page)

3. If $r = 3$, $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L.

$$Q(t) = 25 + 25e^{-0.03t}$$

$$2\% \times 25 = 0.5 \Rightarrow Q(t) = 25.5.$$

$$\frac{0.5}{25} = e^{-0.03T} \Rightarrow T = \frac{\ln 50}{0.03} \approx 130.4 \text{ min}$$

4. Also find the flow rate that is required if the value of T is not to exceed 45 min.

$$T = 45. \quad Q_0 = 50. \quad Q(t) = 25.5.$$

$$25.5 = 25 + 25e^{-r45/100}$$

$$r = \frac{100}{45} \ln 50 \approx 8.69 \text{ gal/min}$$

Remark

- Applications of this model include

pollutant in a lake, drug in an organ of the body.

- In a more general model,

flow rates may vary with time

the rate of inflow and outflow may be different.

Example 2. *(An Extension of Example 1: Problem 4)*

A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow.

$$\frac{dQ}{dt} = 3 - \frac{2Q}{200+t}$$

$$\frac{dQ}{dt} + \frac{2Q}{200+t} = 3 \Rightarrow \mu = (t+200)^2$$

$$Q = \frac{1}{\mu}\left[\int_{t_0}^{t} \mu \cdot 3\, ds + c\right]$$

$$= \frac{1}{(t+200)^2}\left[\int_{t_0}^{t} 3(s+200)^2\, ds + c\right]$$

$$= \frac{1}{(t+200)^2}\left[(t+200)^3 + c\right]$$

$t = 300$.

$$Q = \frac{1}{500^2}\left[\int_0^{300} 3(s+200)^2\, ds + c\right]$$

$$= \frac{1}{500^2}\left[(500)^3 + c\right]$$

$$Q = 500 + \frac{c}{500^2}$$

$t=0,\ Q=100$

$$\Rightarrow 100 = \frac{1}{(300)^2}\left[300^3 + c\right]$$

$$c = -200(300)^2$$

$$= -1.8 \times 10^7$$