MA 266 Lecture 7

Section 2.3 Modeling with First Order Equations (contd)

Example 1. *(Escape Velocity)*

A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance.

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \quad v(0) = v_0$$

An illustration of a body in the earth's gravitational field

Answer the following questions.

1. Find an expression for the velocity during the ensuing motion.

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{(R+x)} + c$$

$t=0.\ x=0,\ v=v_0,\quad x=0,\quad c = v_0^2/2 - gR.$

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}$$

(Example 1 continues next page)

$$\frac{v^2}{2} = \frac{gR^2}{(X+R)} + \frac{1}{2}v_0^2 - gR$$

when $x = \xi$, $v = 0$, $\Rightarrow 0 = \frac{gR^2}{\xi + R} + \frac{1}{2}v_0^2 - gR$

$\Rightarrow$

2. Find the initial velocity that is required to lift the body to a given maximum altitude ξ above the earth.

$$\xi = \frac{v_0^2 R}{2gR - v_0^2} \Rightarrow \quad v_0 = \sqrt{2gR \frac{\xi}{R+\xi}}$$

$\xi \to \infty$, $V_e = \sqrt{2gR}$

escape velocity

3. Find the least initial velocity for which the body will not return to the earth, and it is called the **escape velocity**.

Example 2. *(Problem 21, if time permit)*

A ball with mass $m = 0.15kg$ is thrown upward with initial velocity $20m/s$ from the roof of a building of $30m$ high. The air resistance force of magnitude $|v|/30$ is directed opposite to the velocity.

1. *Find the maximum height above the ground that the ball reaches.*

2. *Find the time that the ball hit the ground.*

$$\begin{cases} m\frac{dv}{dt} = -mg - \frac{|v|}{30} \quad (a) \\ v(0) = +20\ m/s \end{cases}$$

(1) When maximum height, $v = 0$ $\quad v > 0$

$$(a) \Rightarrow \frac{dv}{dt} + \frac{|v|}{30m} = -g \Rightarrow \frac{dv}{dt} + \frac{v}{30m} = -g$$

$$\frac{d\mu}{dt} = +\frac{\mu}{30m} \Rightarrow \mu = ce^{+\frac{t}{30m}}$$

$$d(ve^{+\frac{t}{30m}}) = d(-ge^{+\frac{t}{30m}})$$

$$v = e^{-\frac{t}{30m}} \int -g e^{+\frac{t}{30m}}\, dt$$

$$v = -30mg + ce^{-t/30m}$$

$$+20 = -30mg + c \Rightarrow c = 30mg + 20$$

$$v = -30mg + (30mg + 20)e^{-t/30m}$$

$$\boxed{T = 30m \ln \frac{30mg + 20}{30mg}} = \cancel{2.725}\ 1.683$$

$$\int_0^T v\, dt$$

$$\cancel{h_{max} = \int -30mgT + 30m(30mg - 20)(e^{T/30m} - 1)}$$

$$\cancel{= -119.952 + 108.45 \times 0.822}$$

$$\cancel{= -119.952 + 90.03}$$

$$h'_{max} = \int_0^T v\, dt$$

$$= -30mgT + (30mg + 20) \cdot (-30m) \cdot (e^{-T/30m} - 1)$$

$$= 15.783$$

$$h_{max} = h'_{max} + 30$$