MA 266 Lecture 8

Section 2.4 Differences Between Linear and Nonlinear Equations

Does every initial value problem have exactly one solution?

For a linear equation $y' + p(t)y = g(t)$, we have the following fundamental theorem

Theorem (linear equation) If the function p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then

there exists a unique solution $y = \phi(t)$ that satisfies the IVP for each t in (α, β)

Remark

- The above theorem asserts both the given initial value problem has a solution and only one solution.
- The solution can be discontinuous or fail to exist only at points where at least one of p and g is discontinuous.

For a nonlinear equation $y' = f(t, y)$, we have the following result.

> **Theorem (nonlinear equation)** Let the function f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem
> $$y' = f(t, y), \quad y(t_0) = y_0$$

Remark

- The conditions stated in the above theorem are sufficient to guarantee the existence of a unique solution in some interval $t_0 - h < t < t_0 + h$
- An important geometrical consequence of the uniqueness parts of these theorems is that the graphs of two solutions cannot intersect each other.

Example 1. *Find an interval in which the initial value problem*

$$ty' + 2y = 4t^2, \qquad y(1) = 2$$

has a unique solution.

$$y' + \left(\frac{2}{t}\right) y = 4t.$$

$p(t) = 2/t$, $g(t) = 4t$.

$p(t)$ is continuous for $t < 0$, or $t > 0$.

The interval $t > 0$ contains the initial point.

Theorem guarantees that Example 1 has a unique solution on $0 < t < \infty$.

$$y = t^2 + \frac{1}{t^2}, \quad t > 0$$

Example 2. *Applying the above theorem to the initial value problem*

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1.$$

What can we conclude about the solution of the equation?

$$f(x) = \frac{3x^2+4x+2}{2(y-1)} \quad ; \quad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2+4x+2}{2(y-1)^2}$$

$f(x)$, $\frac{\partial f}{\partial y}$ are continuous for $y \neq 1$.

A rectangle can be drawn about initial point $(0,-1)$ in which $f(x)$, $\frac{\partial f}{\partial y}$ are continuous $\Rightarrow$ Example 2 has a unique solution in some interval about $x=0$.

Example 3. *(Bernoulli Equations)*

The differential equation of the form

$$y' + p(t)y = q(t)y^n$$

is called a **Bernoulli equation***. How to solve these equations?*

- *If* $n = 0$, *or* $n = 1$, for $n=0$, $y' = -p(t)y + q(t) \Rightarrow y ~~= e^{\int p(t)\,dt}~~$

  for $n=1$, $y' = [q(t) - p(t)]y \Rightarrow y = ce^{\int p(t) - q(t)\,dt}$

- *If* $n > 1$,

  let $v = y^{1-n} \Rightarrow \frac{dv}{dt} = y^{-n}\frac{dy}{dt}$, $y = v^{n-1}$

$$v' \cdot y^n + p(t)y = q(t)y^n$$

$$\Rightarrow v' + p(t)y^{1-n} = q(t)$$

$$v' + p(t)v = q(t)$$

$$\frac{d\mu}{dt} = \mu p(t) \Rightarrow \mu = ce^{\int p(t)\,dt}$$

$$\frac{d(\mu v)}{dt} = \mu q(t)$$

$$y = \left(\frac{1}{\mu}\int \mu q\,dt\right)^{n-1} \Leftarrow v = \frac{1}{\mu}\int \mu q(t)\,dt$$

28. $t^2 y' + 2ty - y^3 = 0$

$$y' + \frac{2}{t} y - \frac{y^3}{t^2} = 0$$

Let $v = y^{1-3} = y^{-2}$, $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$

$$-\frac{y^3}{2}\frac{dv}{dt} + \frac{2}{t}y - \frac{y^3}{t^2} = 0$$

$$\frac{du}{dt} - \frac{4}{t}v + \frac{2}{t^2} = 0$$

$$\frac{d\mu}{dt} = -\frac{4}{t}\mu$$

$$\mu = ct^{-4}$$

$$\frac{d(\mu v)}{dt} = \mu \cdot \frac{2}{t^2} = \frac{2}{t^6}$$

$$\mu v = \frac{+2t^{-5}}{5} + c$$

$$\boxed{v = \frac{2t^{-1}}{5} + ct^4}$$

$$\boxed{y = \frac{1}{\pm\sqrt{\frac{2t^{-1}}{5} + ct^4}}}$$

29. $y' = ry - ky^2 \quad r>0,\ k>0$ (population dynamics)

let $v = y^{1-2} = y^{-1}$. $\quad \frac{dv}{dt} = -y^{-2}\frac{dy}{dt}$

$$-y^2\frac{dv}{dt} = ry - ky^2 \Rightarrow \frac{dv}{dt} + rv - k = 0$$

$$\frac{d\mu}{dt} = r\mu \Rightarrow \mu = ce^{rt}$$

$$\frac{d(\mu v)}{dt} = k\mu = ke^{rt}$$

$$\mu v = \frac{k}{r}e^{rt} + c$$

$$\boxed{v = \frac{k}{r} + ce^{-rt}}$$

$$\boxed{y = \frac{1}{\frac{k}{r} + ce^{-rt}}}$$

30. $y' = \varepsilon y - \sigma y^3, \quad \varepsilon>0,\ \sigma>0$ (stability of fluid flow)

let $v = y^{1-3} = y^{-2}$. $\quad \frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$.

$$-\frac{1}{2}y^3\frac{dv}{dt} = \varepsilon y - \sigma y^3 \Rightarrow \frac{dv}{dt} + 2\varepsilon v - 2\sigma = 0$$

$$\frac{d\mu}{dt} = 2\varepsilon \Rightarrow \mu = ce^{2\varepsilon t}$$

$$\frac{d(\mu v)}{dt} = 2\sigma\mu = 2\sigma e^{2\varepsilon t}$$

$$\mu v = \frac{\sigma}{\varepsilon}e^{2\varepsilon t} + c$$

$$\boxed{v = \frac{\sigma}{\varepsilon} + ce^{-2\varepsilon t}}$$

$$\boxed{y = \pm\frac{1}{\sqrt{\frac{\sigma}{\varepsilon} + ce^{-2\varepsilon t}}}}$$

31. $\frac{dy}{dt} = (\Gamma \cos t + T) y - y^3$. Γ, T are constants

Let $v = y^{1-3} = y^{-2}$; $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$.

$$\frac{1}{-2} y^3 \frac{dv}{dt} = (\Gamma \cos t + T) y - y^3$$

$$\frac{dv}{dt} + 2(\Gamma \cos t + T) v - 2 = 0$$

$$\frac{d\mu}{dt} = 2(\Gamma \cos t + T)\mu$$

$$\mu = c e^{2\Gamma \sin t + 2Tt}$$

$$\frac{d(\mu v)}{dt} = 2\mu$$

$$\mu v = \int 2 e^{2\Gamma \sin t + 2Tt} dt$$

$$\boxed{v = \frac{1}{e^{2\Gamma \sin t + 2Tt}} \int 2 e^{2\Gamma \sin t + 2Tt} dt \qquad y = \pm \frac{1}{\sqrt{v}}}$$