MA 266 Lecture 9

Section 2.5 Autonomous Equations and Population Dynamics

A differential equation is called autonomous if it has the form $\frac{dy}{dt} = f(y)$

We know how to solve autonomous equations because they are separable. In this section, we will use geometrical methods to obtain important qualitative information about differential equations of this type.

1. Population Dynamics: Exponential Growth

Let $y = \phi(t)$ be the population of a given species at time t.

$$\begin{cases} \frac{dy}{dt} = ry \\ y(0) = y_0 \end{cases} \Rightarrow y = y_0 e^{rt}$$

Remark

- This model suggests that the population will change exponentially for all time.
- The ideal conditions cannot continue indefinitely, eventually the growth rate will be reduced.

$$\frac{dy}{dt} = (r - ay)y$$

2. Population Dynamics: Logistic Growth

This model takes into account the fact that the growth rate depends on the population.

Replace the constant r by a function

We want the the function $h(y)$ to be close to r when y is small.

$$\frac{dy}{dt} = h(y)y = r\left(1 - \frac{y}{K}\right)y$$

The above equation is called logistic equation. It can be written as $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$

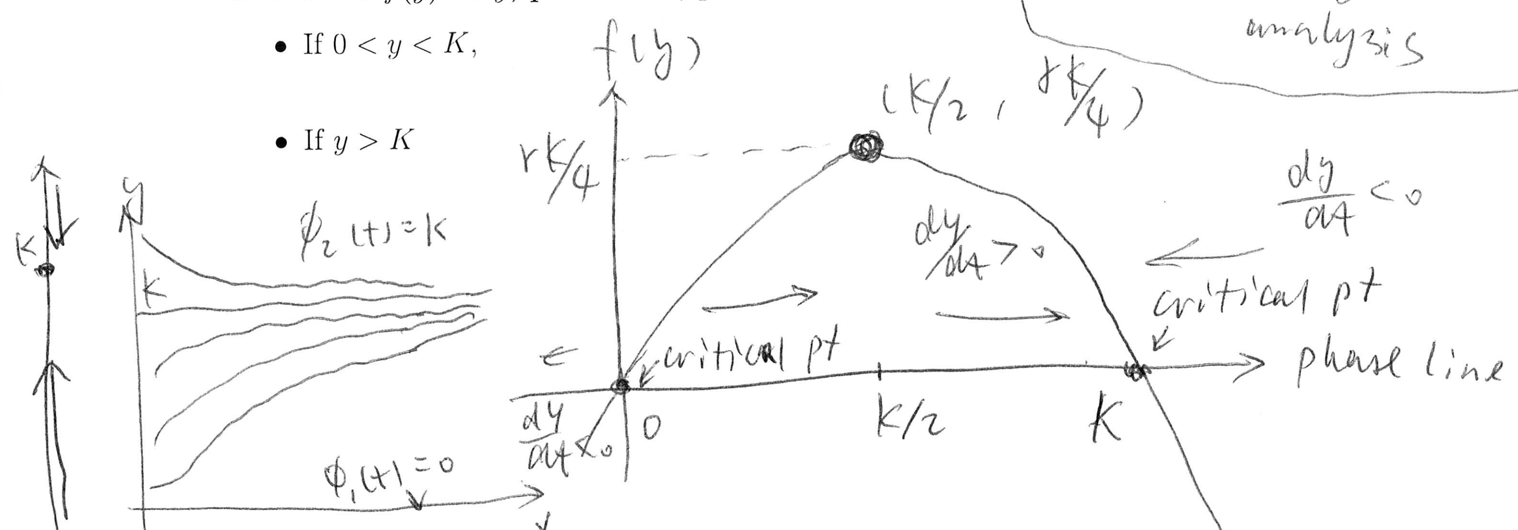
To find constant solution of the equation, we let $r\left(1 - \frac{y}{K}\right)y = 0$

The solutions are called equilibrium solution of the differential equation, or critical points of the function $f(y)$.

3. Plot of $f(y)$ vs y, phase line, plot of solutions

Qualitative geometric analysis

- If $0 < y < K$,
- If $y > K$



plot of $f(y)$ vs y phase line plot of solutions

Remark No solution will

Next, let us analytically solve the initial value problem

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y, \qquad y(0) = y_0.$$

$$\frac{dy}{(1 - y/K)y} = r\,dt$$

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right)dy = r\,dt$$

$$\ln|y| - \ln|1 - y/K| = rt + c$$

If $0 < y_0 < K$: $$\frac{y}{1 - y/K} = ce^{rt}$$

$$y(0) = y_0 \Rightarrow c = y_0/[1 - (y_0/K)]$$

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

For each $y_0 > 0$, the solution approaches the equilibrium solution $y = K$. Hence, we say that

- the constant solution $y = K$ is an asymptotically stable solution
- the constant solution $y = 0$ is an unstable equilibrium solution

Example 1. *(Problem #5) Consider the differential equation*

$$\frac{dy}{dt} = e^{-y} - 1$$

Sketch the graph of $f(y)$ versus y, determine the critical points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions.

$$f(y) = e^{-y} - 1 = 0 \Rightarrow y = 0$$

let $v = e^{-y}$, $\quad \frac{dv}{dt} = -e^{-y}\frac{dy}{dt}$

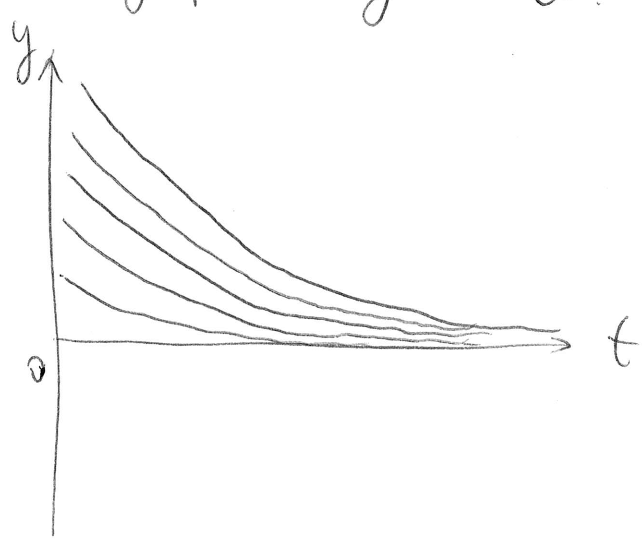
$$-e^{y}\frac{dv}{dt} = v - 1 \Rightarrow \frac{dv}{dt} + v^2 - v = 0$$

$$\Rightarrow \left(\frac{1}{v} - \frac{1}{v-1}\right) dv = dt \Rightarrow v = \frac{e^t}{c_1 + e^t}$$

$$\Rightarrow y = \log(1 - e^{c-t})$$

critical points $y = 0$

$y = 0$ is an asymptotically stable.



Example 2. Logistic model for the growth of the halibut population in Pacific Ocean.

y - total mass of halibut population in t.

$r = 0.71 / \text{year}$, $K = 80.5 \times 10^6$ kg.

$y_0 = 0.25K$

① find biomass 2 years later

$$\frac{y}{K} = \frac{y_0/K}{y_0/K + [1 - y_0/K]e^{-rt}}$$

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + [1 - 0.25]e^{-1.42}} \doteq 0.5797$$

$$\underline{y(2) = 46.7 \times 10^6 \text{ kg}}.$$

② find the time τ for which $y(\tau) = 0.75K$

$$e^{-rt} = \frac{(y_0/K)[1 - (y/K)]}{(y/K)[1 - (y_0/K)]}$$

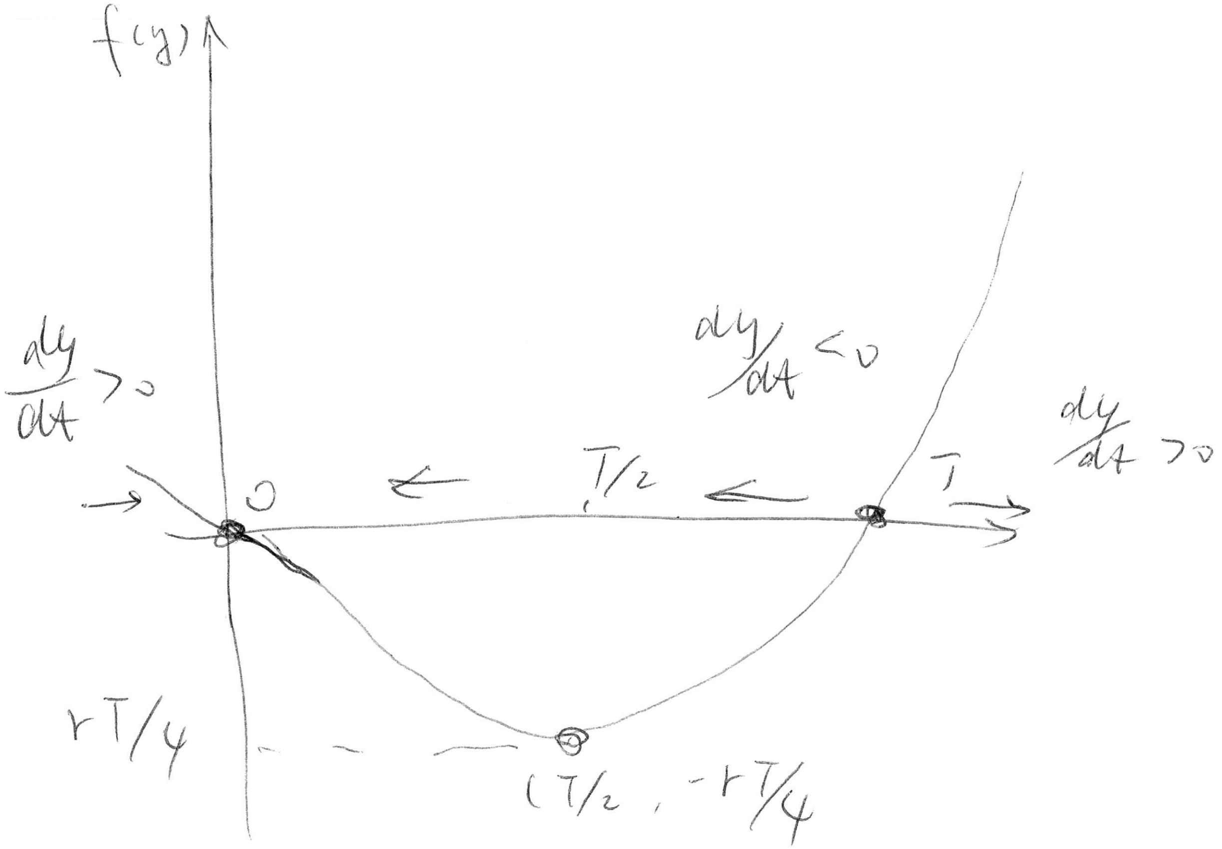
$$\Rightarrow \quad t = -\frac{1}{r} \ln \frac{(y_0/K)[1 - y/K]}{(y/K)(1 - y_0/K)}$$

$$= -\frac{1}{0.71} \ln \frac{0.25 \cdot 0.25}{0.75 \cdot 0.75} = \frac{1}{0.71} \ln 9$$

$$\doteq 3.095 \text{ years}.$$

Example 3. $\frac{dy}{dt} = -r(1 - \frac{y}{T})y$.

where r, and T are positive constants.



$\phi_1(t) = 0$ is an asymptotically stable equilibrium solution and

$\phi_2(t) = T$ is an unstable one.

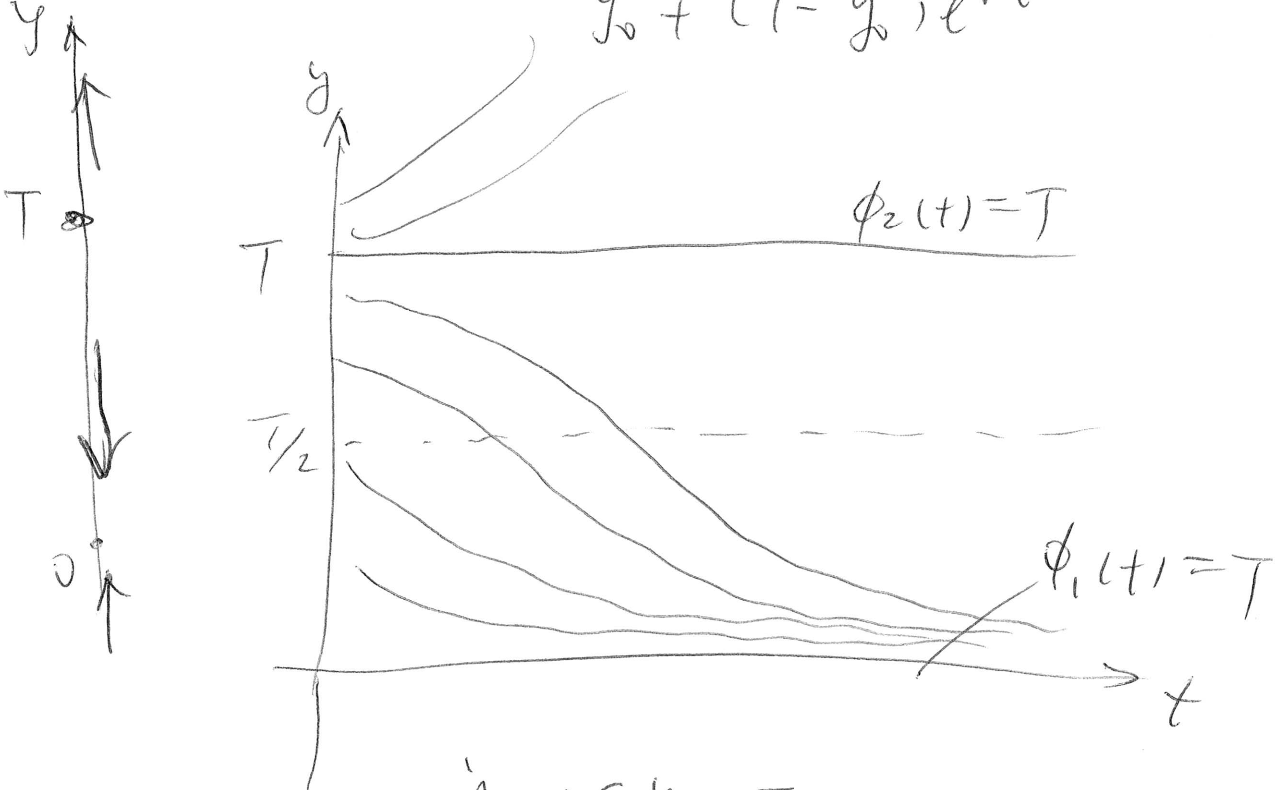
As $t \nearrow$
$$\begin{cases} y \to 0 & \text{if } y_0 < T \\ y \nearrow \infty & \text{if } y_0 > T \\ y = y_0 & \text{if } y_0 = T \end{cases}$$

T is a threshold level, below which growth does not occur.

Example 3:

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad y(0) = y_0$$

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$



if

if $0 < y_0 < T$, $y \to 0$ as $t \to \infty$

if $y_0 > T$ the denominator

$y_0 + (T - y_0)e^{rt^*} = 0$ for certain t^*

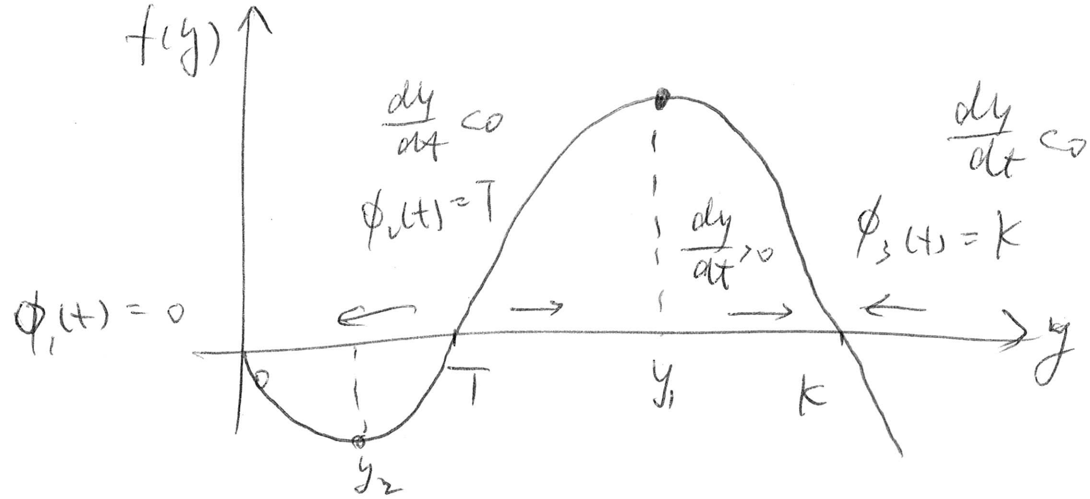
$\Rightarrow t^* = \frac{1}{r} \ln \frac{y_0}{y_0 - T}$. (y_0, T, r)

Example 4. Logistic Growth with a Threshhold

Motivation: unbounded growth does not occur when $y > T$

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y.$$

where $r > 0$, $0 < T < K$.



3 critical points: $y = 0$, $y = T$ and $y = K$.

$\phi_2(t) = T$ is an unstable solution.

$\frac{dy}{dt} > 0$, for $T < y < K$, $y \nearrow$ as $t \uparrow$

$\frac{dy}{dt} < 0$ for $y < T$, $y \searrow$ as $t \uparrow$

$\frac{dy}{dt} < 0$ for $y > T$, $y \searrow$ as $t \uparrow$

$\Rightarrow$ $\phi_3(t) = K$ is an asymptotically stable solution.

$\phi_1(t) = 0$ is an asymptotically stable solution.

Example 4.

The maximum and minimum points, y_1, y_2 can be obtained by differentiating RHS of

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y = f(x)$$

with respect to y, setting the result equal to zero

$$\frac{df}{dy} = 0 \Rightarrow y_1 = \frac{\left(K + T + \sqrt{K^2 - KT + T^2}\right)}{3}$$

$$y_2 = \frac{\left(K + T - \sqrt{K^2 - KT + T^2}\right)}{3}$$